# Hedging costs for two large investors

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**Abstract**

We consider the cost of hedging contingent claims in a financial market where the trades of two large investors can move market prices. We provide a characterization of the minimal hedging costs in terms of associated stochastic control problems. We also prove that the minimal hedging cost is a viscosity solution of a corresponding dynamic programming equation in the case of a Markov market model.

**Key words:** large investor, hedging cost, dynamic programming equation.

## 1 Introduction

Our concern is to examine the cost of hedging contingent claims in a financial market where the trades of two large investors can move market prices, and the purpose of this paper is to provide a characterization of the minimal hedging costs in terms of associated stochastic control problems.

### 1.1 General large investor problem

Let $T > 0$ be a finite time horizon and $\{W(t),\, 0 \leq t \leq T\}$ a standard $d$-dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, endowed with a filtration $\mathbb{F} = \{\mathcal{F}_t,\, 0 \leq t \leq T\}$ which is the $\mathbb{P}$-augmentation of the filtration generated by the Brownian motion $W$. Let $\mathcal{P}$ denote the set of all $\mathbf{R}^n$-valued, $\mathbb{F}$-progressively measurable processes $p(\cdot)$ such that $\int_0^T |p(t)|^2\, dt < \infty$ a.s. Here $n \leq d$.

### 1.1.a *Price dynamics in presence of large investors*

We assume that there are two large investors $\mathbb{I}^k$ $(k=0,1)$ in a financial market where one bank account and $n$ stocks are traded continuously up to the time $T$. For each $k=0,1$, let $\varPi^k \subset \mathcal{P}$ be a set of admissible trading strategies of the investor $\mathbb{I}^k$, and we write $\pi=(\pi^0,\pi^1)\in \varPi := \varPi^0 \times \varPi^1$. We assume $0 \in \varPi^0 \cap \varPi^1$.

Consider a model for price fluctuations as follows: If the investor $\mathbb{I}^k$ starts at time 0 with an initial capital $x_k \in \mathbf{R}$ and holds $\pi_j^k(t)$ shares of the $j$-th stock at time $t\in[0,T]$, $j=1,\ldots,n$, $k=0,1$, then the price processes $B^\pi(\cdot)$ of the bank account and $\widehat{S}^\pi(\cdot)$ of the stocks evolve according to the stochastic differential equation (SDE, for short)

$$
\begin{aligned}
&dB(t)=B(t)r^\pi(t)dt, \qquad B(0)=1,\\
&d\widehat{S}(t)=diag[\widehat{S}(t)]\left\{b^\pi(t)dt+\sigma^\pi(t)^\top dW(t)\right\}, \qquad \widehat{S}(0)=s\in(0,\infty)^n,
\end{aligned}
$$

and the discounted wealth process $X_k^{x_k,\pi}(\cdot)$ of the investor $\mathbb{I}^k$ is given as

$$
X_k(t)=x_k+\int_0^t \pi^k(u)^\top dS^\pi(u), \qquad t\in[0,T], \qquad k=0,1, \tag{1.1}
$$

where $\top$ denotes the transpose operation; $diag[s]$ is the $n\times n$-diagonal matrix with diagonal elements $s_1,\ldots,s_n$; $S^\pi(\cdot) := B^\pi(\cdot)^{-1}\widehat{S}^\pi(\cdot)$ is the discounted price process of stocks; $\{r^\pi(t),\ 0\le t\le T\}$, $\{b^\pi(t)=(b_1^\pi(t),\ldots,b_n^\pi(t))^\top,\ 0\le t\le T\}$ and $\{\sigma^\pi(t)=(\sigma_1^\pi(t)\cdots\sigma_n^\pi(t)),\ 0\le t\le T\}$ are bounded $\mathbb{F}$-progressively measurable processes taking values in $\mathbf{R}_+$, $\mathbf{R}^n$ and $\mathbf{R}^d\otimes\mathbf{R}^n$, respectively. Here the superscript $\pi$ means that the process $h^\pi(t,\omega)$ $(h=r,b,\sigma$, for instance) with the superscript $\pi$ depends on the path $\{\pi(u,\omega),\ 0\le u\le t\}$ for a.e. $(t,\omega)\in[0,T]\times\Omega$ and $\pi\in\varPi$. Therefore the price dynamics are influenced by the actions of the investors. It is for this reason that $\mathbb{I}^k$ is called the *large investor*. We also remark that the integral in (1.1) is well-defined by means of $\int_0^T \left|\pi^k(t)\right|^2 dt<\infty$ a.s. and of the boundedness of the coefficients of market.

### 1.1.b *Contingent claim and minimal hedging costs*

A contingent claim $\{B^\pi C^\pi, \mathcal{T}^\pi\}$ consists of an $\mathbb{F}$-adapted, non-negative process $\{C^\pi(t), 0\le t\le T\}$ and some class $\mathcal{T}^\pi$ of $\mathbb{F}$-stopping times. We assume $T\in\mathcal{T}^\pi$. Let us consider now the following situation: At time $t=0$, $\mathbb{I}^0$ and $\mathbb{I}^1$ enter into an agreement. The seller $\mathbb{I}^0$ agrees to provide the buyer $\mathbb{I}^1$ with the random payment $B^\pi(\tau(\omega),\omega)C^\pi(\tau(\omega),\omega)$ at time $t=\tau(\omega)$, where $\tau$ is an element of $\mathcal{T}^\pi$ and at the disposal of the buyer.

The objective of $\mathbb{I}^0$ is to find a portfolio strategy $\pi^0\in\varPi^0$ which is chosen according to a trading strategy $\pi^1\in\varPi^1$ of $\mathbb{I}^1$ and enables him to fulfill his obligation whenever $\mathbb{I}^1$ decides to ask for the payment. Hence the upper hedging cost $h_{up}$ is defined as

$$
h_{up} := \inf\left\{ x_0\ge 0 \;\middle|\; \begin{array}{l} \forall\pi^1\in\varPi^1,\ \exists\pi^0\in\varPi^0 \quad \text{s.t.}\\ \quad X_0^{x_0,\pi}(\tau)\ge C^\pi(\tau) \quad \text{a.s.}, \qquad \forall\tau\in\mathcal{T}^\pi. \end{array}\right\}. \tag{1.2}
$$

On the other hand, the objective of $\mathbb{I}^1$ is to find both a portfolio strategy $\pi^1\in\varPi^1$ and exercise time $\tau\in\mathcal{T}^\pi$, which are chosen according to a trading strategy $\pi^0\in\varPi^0$ of

$\mathbb{I}^0$, such that the payment he receives at $t = \tau(\omega)$ allows him to cover the debt $-x_1$ be incurred at $t = 0$ by purchasing the contingent claim. Therefore the lower hedging cost $h_{low}$ is defined as

$$h_{low} := \sup\left\{ x_1 \geq 0 \;\middle|\; \begin{array}{c} \forall \pi^0 \in \mathit{\Pi}^0,\ \exists \pi^1 \in \mathit{\Pi}^1,\ \exists \tau \in \mathcal{T}^\pi \quad \text{s.t.} \\ X_1^{-x_1,\pi}(\tau) + C^\pi(\tau) \geq 0 \quad \text{a.s.} \end{array} \right\}. \tag{1.3}$$

### 1.2 Existent studies dealing with analogous models

In a standard market of a small investor model (the coefficients $r, b$ and $\sigma$ do not depend on $\pi$), there exists an $\mathbb{F}$-progressively measurable process $\theta : [0, T] \times \Omega \to \mathbf{R}^d$ such that

$$b(t,\omega) - r(t,\omega)\mathbf{1}_n = -\sigma(t,\omega)^\top \theta(t,\omega) \qquad \text{a.e. } (t,\omega) \in [0,T] \times \Omega \tag{1.4}$$

and the stochastic exponential process

$$Z_\theta(t) := \exp\left\{ \int_0^t \theta(u)^\top dW(u) - \frac{1}{2}\int_0^t |\theta(u)|^2 du \right\}, \qquad 0 \leq t \leq T \tag{1.5}$$

is a martingale, where $\mathbf{1}_n = (1, \ldots, 1)^\top \in \mathbf{R}^n$. Further, if the standard market is complete, by the martingale representation theorem and the Bayes' rule, we then have

$$\mathbb{E}\left[ \frac{Z_\theta(T)}{Z_\theta(t)} C(T) \middle| \mathcal{F}_t \right] = \mathbb{E}\left[Z_\theta(T)C(T)\right] + \int_0^t \pi^0(u)^\top dS(u), \qquad t \in [0,T] \tag{1.6}$$

for some hedging portfolio $\pi^0$. Therefore the minimal hedging cost[1] of European contingent claim $\{B(T)C(T), \{T\}\}$ is given by $h_{up} = h_{low} = \mathbb{E}[Z_\theta(T)C(T)]$.

In some special cases, we can also use the martingale duality approach to study the replication of European contingent claims by the large investor. Cuoco & Liu[6] has provided the dual formulation for the case that $r^\pi, \sigma^\pi$ and $C^\pi$ are independent of the trading strategy $\pi$, and $b^\pi$ satisfies

$$q^0(t,\omega)^\top b^\pi(t,\omega) = q^0(t,\omega)^\top \mu(t,\omega) + h(t, q^0(t,\omega), \omega) \qquad \text{a.e. } (t,\omega) \in [0,T] \times \Omega,$$

where $q^0(t) = X_0(t)^{-1} diag[S(t)]\pi^0(t)$, $\mu$ is a bounded $\mathbb{F}$-progressively measurable process taking values in $\mathbf{R}^n$, and a function $h(t,q,\omega)$ on $[0,T] \times \mathbf{R}^n \times \Omega$ satisfies: $h(\cdot, q, \cdot)$ is an optional process for each $q \in \mathbf{R}^n$; $h(t,\cdot,\omega)$ is Lipschitz uniformly in $(t,\omega) \in [0,T] \times \Omega$; $h(t,\cdot,\omega)$ is concave and upper semicontinuous for all $(t,\omega) \in [0,T] \times \Omega$; $h(t,0,\omega) = 0$ for all $(t,\omega) \in [0,T] \times \Omega$. Bank & Baum[2] dealt with a general semimartingale market model with a *single* large investor. They presented a characterization of the upper hedging cost for European contingent claims in terms of an associated stochastic control problem under the condition that (1.4) was satisfied for some process $\theta$ which did not depend on the large investor's position $\pi^0$ despite of the dependence of $r, b$ and $\sigma$ upon $\pi^0$.

In the case of the general large investor model, however, it is difficult to use the

[1] For a standard asset pricing theory, see the usual textbooks, e.g. Duffie[11], Karatzas[15] and Karatzas & Shreve[17].

martingale duality approach in order to show the existence of a portfolio $\pi^0$ satisfying (1.6) because of the dependence of $\theta, C$ and $S$ upon $\pi$. Hence the martingale duality approach has not been successful to solve the general large investor problem. Therefore the previous studies have provided several treatments of this problem which avoid the passage from the dual formulation. These studies dealt with Markov market models with a *single* large investor as follows:

(i) Cvitanić & Ma[9]: For $h = b, \sigma$,

$$h^\pi(t) = h(t, S(t), \pi^0(t), X_0(t)), \qquad r^\pi(t) = r(t, diag[S(t)]\pi^0(t), X_0(t)).$$

(ii) Soner & Touzi[24]: For $h = b, \sigma$ and $q^0(t) = X_0(t)^{-1} diag[S(t)]\pi^0(t)$,

$$h^\pi(t) = h(t, S(t), q^0(t)), \qquad r^\pi(t) \equiv 1.$$

(iii) Frey[12]: In one-dimensional case ($n = d = 1$),

$$S^\pi(t) = \psi(t, Z_\eta(t), \pi^0(t)), \qquad r^\pi(t) \equiv 1, \qquad \pi^0(t) = \phi(t, Z_\eta(t)),$$

where $\psi$ is a smooth reaction function, the stochastic exponential $Z_\eta$ defined as (1.5) with a constant $\eta$ is a fundamental state variable process, and the trading strategy $\phi$ is selected from among smooth functions. In Platen & Schweizer[20] and Frey & Stremme[13] the state variable $Z_\eta$ and the reaction function $\psi$ have been obtained from equilibrium considerations.

Cvitanić & Ma [9] characterized the cost and portfolio of hedging European option $B(T)C(T) = g(S(T))$ as a solution of a forward-backward SDE corresponding to their Markov model, and proved the existence and uniqueness of the solution of this equation under regularity conditions on $r, b, \sigma$ and $g$. Frey[12] characterized the hedging portfolio $\phi$ of European option $C(T) = g(S(T))$ as a solution of an associated quasi-linear partial differential equation and provided results on existence and uniqueness of the solution to this equation under regularity conditions on $\psi$ and $g$. Soner & Touzi[24] used a new dynamic programming principle established in Soner & Touzi[22] to characterize the minimal hedging cost for European option $C(T) = g(S(T))$ as a viscosity solution of a corresponding dynamic programming equation under suitable conditions on $b, \sigma$ and $g$.

Since $r, b$ and $\sigma$ in our model do not depend on the value of the large investor's wealth $X_0$, our model does not include those of Cvitanić & Ma[9] and Soner & Touzi[24]. As seen in Appendix B, however, we can apply our approach to the study of the replication in the model of Soner & Touzi[24], and we can treat Example 5.1 of Cvitanić & Ma[9] in our framework (see Remark B.2 below). Extending the set of admissible portfolios in the model of Frey[12] to the set of controlled semimartingales

$$d\pi^0(t) = \alpha(t)dt + \beta(t)dW(t) \qquad \text{(where } \alpha \text{ and } \beta \text{ are controls)},$$

we also have the application of our approach to the study of the replication in the model of Frey[12].

The remainder of this paper is organized in the following way: In the next section we characterize the minimal hedging costs in terms of associated stochastic control problems. In §3, we derive a corresponding dynamic programming equation from the representation obtained in §2, and characterize the minimal hedging cost as a viscosity solution of this equation in the case of a Markov market model. The proofs of assertions stated in §2 and §3 are given in §4 and §5, respectively. In Appendix A, we mention briefly the points of an absence of arbitrage opportunity in our market model.

## 2 Main result

In order to characterize the hedging costs in terms of the stochastic control problems, we shall introduce the notion of the change of measure. Let $\mathcal{D}_m$ be the class of all $\mathbf{R}^d$-valued, $\mathbb{F}$-progressively measurable processes $\nu(\cdot)$ such that $|\nu(t,\omega)| \le m$ a.e., and $\mathcal{D} := \bigcup_{m=1}^{\infty} \mathcal{D}_m$. Then the stochastic exponential process

$$Z_\nu(t) := \exp\left\{ \int_0^t \nu(u)^\top dW(u) - \frac{1}{2}\int_0^t |\nu(u)|^2 du \right\}, \qquad 0 \le t \le T$$

is a martingale for each $\nu \in \mathcal{D}$, and

$$\mathbb{P}_\nu(\Lambda) := \mathbb{E}\left[Z_\nu(T)\mathbb{1}_\Lambda\right], \qquad \Lambda \in \mathcal{F}_T$$

is a probability measure, where $\mathbb{1}$ is the indicator function. For the change of measure,we note the Bayes' rule[2]: For every $\mathcal{F}_T$-measurable random variable $Y \ge 0$ a.s.,

$$\mathbb{E}_\nu[Y|\mathcal{F}_t] = \mathbb{E}\left[ \frac{Z_\nu(T)}{Z_\nu(t)} Y \middle| \mathcal{F}_t \right], \qquad \nu \in \mathcal{D},$$

where $\mathbb{E}_\nu$ denotes the expectation operator under $\mathbb{P}_\nu$.

When the seller $\mathbb{I}^0$ receives the amount $x > h_{up}$ from the buyer $\mathbb{I}^1$, he can cover his obligation at any time $\tau \in \mathcal{T}^\pi$ *without risk*, i.e.

$$\sup_{\pi^1\in\Pi^1} \inf_{\pi^0\in\Pi^0} \sup_{\tau\in\mathcal{T}^\pi} \sup_{\nu\in\mathcal{D}} \mathbb{E}_\nu\left[(C^\pi(\tau) - X_0^{x,\pi}(\tau))^+\right] = 0,$$

where $a^+ := a \vee 0 = \max\{a, 0\}$. Formally, we calculate

$$\begin{aligned}
0 &= \sup_{\pi^1\in\Pi^1} \inf_{\pi^0\in\Pi^0} \sup_{\tau\in\mathcal{T}^\pi} \sup_{\nu\in\mathcal{D}} \mathbb{E}_\nu\left[(C^\pi(\tau) - X_0^{x,\pi}(\tau))^+\right] \\
&\text{``}\overset{?}{=}\text{''} \sup_{\pi^1\in\Pi^1} \inf_{\pi^0\in\Pi^0} \sup_{\tau\in\mathcal{T}^\pi} \sup_{\nu\in\mathcal{D}} \mathbb{E}_\nu\left[C^\pi(\tau) - X_0^{x,\pi}(\tau)\right] \\
&= \sup_{\pi^1\in\Pi^1} \inf_{\pi^0\in\Pi^0} \sup_{\tau\in\mathcal{T}^\pi} \sup_{\nu\in\mathcal{D}} \mathbb{E}_\nu\left[C^\pi(\tau) - X_0^{0,\pi}(\tau)\right] - x \\
&\text{``}\overset{?}{=}\text{''} \sup_{\pi^1\in\Pi^1} \inf_{\pi^0\in\Pi^0} \sup_{\tau\in\mathcal{T}^\pi} \sup_{\nu\in\mathcal{D}} \mathbb{E}_\nu\left[\left(C^\pi(\tau) - X_0^{0,\pi}(\tau)\right)^+\right] - x,
\end{aligned}$$

[2]See Lemma 3.5.3 in Karatzas & Shreve[16] and Exercise 0.3.6 in Karatzas[15].

and letting $x \downarrow h_{up}$, we conjecture

$$h_{up} = \sup_{\pi^1 \in \Pi^1} \inf_{\pi^0 \in \Pi^0} \sup_{\tau \in \mathcal{T}^\pi} \sup_{\nu \in \mathcal{D}} \mathbb{E}_\nu \left[ \left( C^\pi(\tau) - X_0^{0,\pi}(\tau) \right)^+ \right]. \tag{2.1}$$

Indeed, we have our main result as follows:

**Theorem 2.1** *The upper hedging cost $h_{up}$ is expressed as* (2.1) *and the lower hedging cost $h_{low}$ is given as*

$$h_{low} = \lim_{m \to \infty} \inf_{\pi^0 \in \Pi^0} \sup_{\pi^1 \in \Pi^1} \sup_{\tau \in \mathcal{T}^\pi} \inf_{\nu \in \mathcal{D}} \mathbb{E}_\nu \left[ \left( C^\pi(\tau) + X_1^{0,\pi}(\tau) \right) \wedge m \right], \tag{2.2}$$

*where $a \wedge b := \min\{a, b\}$. Moreover,*

(i) *If* $\mathbb{E}[|X_0^{0,\pi}(\tau)|^p] < \infty$ *for any $\pi \in \Pi$, $\tau \in \mathcal{T}^\pi$ and some constant $p = p(\pi, \tau) > 1$, then*

$$h_{up} = \left( \sup_{\pi^1 \in \Pi^1} \inf_{\pi^0 \in \Pi^0} \sup_{\tau \in \mathcal{T}^\pi} \sup_{\nu \in \mathcal{D}} \mathbb{E}_\nu \left[ C^\pi(\tau) - X_0^{0,\pi}(\tau) \right] \right)^+. \tag{2.3}$$

(ii) *If* $\mathbb{E}[C^\pi(\tau)^p + |X_1^{0,\pi}(\tau)|^p] < \infty$ *for any $\pi \in \Pi$, $\tau \in \mathcal{T}^\pi$ and some constant $p = p(\pi, \tau) > 1$, then*

$$h_{low} = \inf_{\pi^0 \in \Pi^0} \sup_{\pi^1 \in \Pi^1} \sup_{\tau \in \mathcal{T}^\pi} \inf_{\nu \in \mathcal{D}} \mathbb{E}_\nu \left[ C^\pi(\tau) + X_1^{0,\pi}(\tau) \right]. \tag{2.4}$$

*Proof* The proof is given in §4. □

**Remark 2.2** When we defer to the suggestion of Bank & Baum[2] and Çetin *et al.*[4], the discounted wealth process $X_k^\pi$ should be replaced by

$$\widetilde{X}_k^\pi(t) = X_k^\pi(t) - \frac{L^\pi(t)}{B^\pi(t)}, \qquad 0 \le t \le T,$$

where $\{L^\pi(t),\ 0 \le t \le T\}$ is a right-continuous, $\mathbb{F}$-adapted increasing process with $L^\pi(0) = 0$. Here $L^\pi(t)$ has the interpretation of the cumulative cost of the liquidity risk up to time $t \in [0, T]$. As seen in the proof stage, however, it is clear that if we replace $X_k^\pi$ with $\widetilde{X}_k^\pi$ in the equations (2.1)-(2.4), the assertions in the previous theorem remain to be true without additional assumptions on $L^\pi$.

In order to obtain further sharp results, we are now in a position to make some assumptions:

**Assumption 2.3** (i) For all $\pi \in \Pi$ there exists $\theta^\pi \in \mathcal{D}$ such that

$$b^\pi(t, \omega) - r^\pi(t, \omega)\mathbf{1}_n = -\sigma^\pi(t, \omega)^\top \theta^\pi(t, \omega) \qquad \text{a.e. } (t, \omega) \in [0, T] \times \Omega, \tag{2.5}$$

where $\mathbf{1}_n = (1, \ldots, 1)^\top \in \mathbf{R}^n$.

(ii) For all $\pi \in \Pi$ there exists a constant $p > 1$ such that

$$\mathbb{E}\left[\int_0^T |\pi(t)|^{2p}\, dt\right] < \infty. \tag{2.6}$$

In the case of a small investor model, the process $-\theta$ of (2.5) is called the market price of risk process and the risk-neutral equivalent martingale measure $\mathbb{P}_\theta$ plays an important role for the pricing theory, as stated in §1.2. Moreover the conditions (2.5)-(2.6) guarantees that there is no arbitrage opportunity in a standard market of the small investor model. Therefore it seems natural to assume (2.5)-(2.6).

**Corollary 2.4** *Under Assumption 2.3, we have*

$$h_{up} = \sup_{\pi^1 \in \Pi^1} \inf_{\pi^0 \in \Pi^0} \sup_{\tau \in \mathcal{T}^\pi} \sup_{\nu \in \mathcal{D}} \mathbb{E}_\nu \left[C^\pi(\tau) - X_0^{0,\pi}(\tau)\right]. \tag{2.7}$$

*Moreover, if* $\mathbb{E}[C^\pi(\tau)^p] < \infty$ *for any* $\pi \in \Pi$, $\tau \in \mathcal{T}^\pi$ *and some constant* $p = p(\pi, \tau) > 1$, *then we have the expression* (2.4) *of the lower hedging cost.*

*Proof* We may assume $p < 2$ in (2.6). Let $\pi \in \Pi$ be arbitrary. Since the process $\theta^\pi \in \mathcal{D}$ is bounded a.e., we have (2.3)-(2.4) if we can prove

$$\mathbb{E}_{\theta^\pi}\left[\sup_{0\le t\le T} \left|X_k^{0,\pi}(t)\right|^p\right] < \infty, \qquad k = 0, 1. \tag{2.8}$$

Furthermore the above estimate guarantees that $\{X_k^{0,\pi}(t),\ 0 \le t \le T\}$ is a martingale under $\mathbb{P}_{\theta^\pi}$ for all $\pi \in \Pi$ and $k = 0, 1$. Thus (2.7) is deduced from (2.3). Hence it is enough to prove (2.8). Indeed, the Burkholder-Davis-Gundy inequalities and the Hölder inequality yield

$$\begin{aligned}
\mathbb{E}_{\theta^\pi}\left[\sup_{0\le t\le T} \left|X_k^{0,\pi}(t)\right|^p\right] &\le c_p\, \mathbb{E}_{\theta^\pi}\left[\left(\int_0^T \left|\sigma^\pi(t) diag[S^\pi(t)]\pi^k(t)\right|^2 dt\right)^{\frac{p}{2}}\right] \\
&\le c_p \|\sigma^\pi\|^p\, \mathbb{E}_{\theta^\pi}\left[\left(\int_0^T \left|\pi^k(t)\right|^2 |S^\pi(t)|^2\, dt\right)^{\frac{p}{2}}\right] \\
&\le c_p \|\sigma^\pi\|^p\, \mathbb{E}_{\theta^\pi}\left[\left(\int_0^T \left|\pi^k(t)\right|^{2p} dt\right)^{\frac{p}{2}}\right]^{\frac{1}{p}} \cdot \mathbb{E}_{\theta^\pi}\left[\left(\int_0^T |S^\pi(t)|^{2q}\, dt\right)^{\frac{p}{2}}\right]^{\frac{1}{q}} \\
&\le c_p \|\sigma^\pi\|^p\, \mathbb{E}\left[Z_{\theta^\pi}(T)^{q_1}\right]^{\frac{1}{pq_1}} \cdot \mathbb{E}\left[\int_0^T \left|\pi^k(t)\right|^{2p} dt\right]^{\frac{1}{2}} \cdot \mathbb{E}_{\theta^\pi}\left[\int_0^T |S^\pi(t)|^{2q}\, dt\right]^{\frac{p}{2q}}
\end{aligned}$$

for $q = p/(p-1)$, $q_1 = 2/(2-p)$ and some constant $c_p > 0$, where $\|\sigma^\pi\| := \sup\{|\sigma_i^\pi(t,\omega)| : (t,\omega) \in [0,T] \times \Omega,\ 1 \le i \le n\}$. By the standard arguments, we also have

$$\mathbb{E}_{\theta^\pi}\left[\int_0^T |S^\pi(t)|^{2q}\, dt\right] \le |s|^{2q} T e^{q(2q-1)\|\sigma^\pi\|^2 T}.$$

Therefore we obtain (2.8). □

**Corollary 2.5** *Suppose that Assumption 2.3 holds. If $B^\pi$, $S^\pi$, $C^\pi$ and $\mathcal{T}^\pi$ are independent of $\pi^1$ for all $\pi = (\pi^0, \pi^1) \in \Pi$, then $h_{low} \leq h_{up}$.*

Proof As seen above, the process $\{X_k^{0,\pi}(t),\ 0 \leq t \leq T\}$ is a $\mathbb{P}_{\theta^\pi}$-martingale for all $\pi \in \Pi$ and $k = 0, 1$, under Assumption 2.3. Hence it follows from (2.1)-(2.2) that

$$h_{low} \leq \inf_{\pi^0 \in \Pi^0} \sup_{\tau \in \mathcal{T}^\pi} \mathbb{E}_{\theta^\pi}\left[C^\pi(\tau)\right] \leq h_{up}.$$

□

# 3 Dynamic programming equations

## 3.1 Markov market model

In order to adapt the arguments developed by Soner & Touzi[22]-[24] and Bensoussan *et al.*[3] to our large investors model, we now focus on the Markov case:

$$h^\pi(t) = h(t, B(t), S(t), \pi(t)), \qquad \text{for } h = r, b, \sigma,$$

where $r, b$ and $\sigma$ are $\mathbf{R}_+, \mathbf{R}^n$ and $\mathbf{R}^d \otimes \mathbf{R}^n$-valued, bounded functions defined on $[0,T] \times \mathbf{R}_+ \times \mathbf{R}_+^n \times \mathbf{R}^{2n}$. We further assume that $r, b$ and $\sigma$ are Lipschitz functions in the $(\beta, s, \pi)$ variable, uniformly in $t$. We consider the special case of European contingent claim:

$$C^\pi(T) = g(B(T), S(T)) \qquad \text{and} \qquad \mathcal{T}^\pi = \{T\},$$

where a non-negative function $g$ on $(0,\infty) \times \mathbf{R}_+^n$ satisfies the polynomial growth condition:

$$g(\beta, s) \leq c_0(\beta^{-l} + \beta^l + |s|^l), \qquad (\beta, s) \in (0, \infty) \times \mathbf{R}_+^n$$

for certain constants $c_0, l > 0$.

Let $K_j \subset \mathbf{R}^n$ $(j = 0, 1)$ be compact convex subsets which contain the origin. We assume that $\Pi^j$ is the set of all processes $\pi \in \mathcal{P}$ such that $\pi(t, \omega) \in K_j$ a.e. for $j = 0, 1$. Let $\delta_j$ denote the support function $\delta_j(q) := \sup_{p \in K_j}(p^\top q)$, $q \in \mathbf{R}^n$, $j = 0, 1$. Define

$$\begin{aligned}
\mathcal{H}_j(p) &:= \inf\Big\{\delta_j(q) - q^\top p \ : \ |q| = 1\Big\}, \qquad p \in \mathbf{R}^n, \quad j = 0, 1, \\
\widehat{h}_j(\beta, s) &:= \sup_{q \in \mathbf{R}_+^n}\Big\{h(\beta, q) - \delta_j(q - s)\Big\}, \qquad (\beta, s) \in (0, \infty) \times \mathbf{R}_+^n, \quad j = 0, 1,
\end{aligned}$$

for each function $h : (0, \infty) \times \mathbf{R}_+^n \to \mathbf{R}$. It is well known that the support function $\delta_j$ is non-negative, convex and positively homogeneous, and

$$\text{“ } p \in K_j \quad \Leftrightarrow \quad \mathcal{H}_j(p) \geq 0 \text{ ”} \qquad \text{and} \qquad \text{“ } p \in \operatorname{int} K_j \quad \Leftrightarrow \quad \mathcal{H}_j(p) > 0 \text{ ”} \tag{3.1}$$

for $j = 0, 1$. Furthermore we assume that $\sigma$ satisfies the uniform ellipticity condition:

$$|\sigma(y, \pi)\xi| \geq c|\xi|, \qquad y \in [0, T] \times \mathbf{R}_+^{n+1}, \ \pi \in K_0 \times K_1, \ \xi \in \mathbf{R}^n, \tag{3.2}$$

for some constant $c > 0$. This condition guarantees that there exists a bounded function

$\theta : [0,T] \times \mathbf{R}_+^{n+1} \times (K_0 \times K_1) \to \mathbf{R}^d$ such that

$$-\sigma(y,\pi)^\top \theta(y,\pi) = b(y,\pi) - r(y,\pi)\mathbf{1}_n, \tag{3.3}$$

and hence the condition (2.5) holds.

### 3.2 Stochastic control problems and dynamic programming equations

Thanks to Girsanov's theorem, we have

$$\mathbb{E}_{\nu+\theta}\left[X_j^{0,\pi}(T)\right] = \mathbb{E}_{\nu+\theta}\left[\int_0^T \pi^j(u)^\top diag[S(u)]\sigma(Y(u),\pi(u))^\top \nu(u)du\right]$$

for $\pi \in \Pi$, $\nu \in \mathcal{D}$, $j = 0,1$, where $Y(u) := (u, B(u), S(u))^\top$ and $\theta(u) = \theta(Y(u),\pi(u))$. From (2.7) and (2.4), therefore, we can derive the stochastic control problems:

$$U(y) := \sup_{\pi^1\in\Pi^1}\inf_{\pi^0\in\Pi^0}\sup_{\nu\in\mathcal{D}}\mathbb{E}^y\left[g(B^\pi(T), S^{\pi,\nu}(T)) - \int_t^T \pi^0(u)^\top diag[S^{\pi,\nu}(u)]\sigma(a(u))^\top\nu(u)du\right], \tag{3.4}$$

$$\widetilde{L}(y) := \inf_{\pi^0\in\Pi^0}\sup_{\pi^1\in\Pi^1}\inf_{\nu\in\mathcal{D}}\mathbb{E}^y\left[g(B^\pi(T), S^{\pi,\nu}(T)) + \int_t^T \pi^1(u)^\top diag[S^{\pi,\nu}(u)]\sigma(a(u))^\top\nu(u)du\right], \tag{3.5}$$

for $y = (t,\beta,s) \in [0,T]\times(0,\infty)\times(0,\infty)^n$, where $a(u) = (Y^{\pi,\nu}(u),\pi(u))^\top$, $S^{\pi,\nu}$ is a unique solution of the equation

$$dS(u) = diag[S(u)]\sigma(a(u))^\top\{\nu(u)du + dW(u)\}, \qquad t \le u \le T, \tag{3.6}$$

and the suffix $y = (t,\beta,s)$ of $\mathbb{E}$ means that we have specified the data $(B^\pi(t), S^{\pi,\nu}(t)) = (\beta, s)$.

Then, since $\{\sigma_i(y,\pi)\}_i$ is linearly independent by means of (3.2), the dynamic programming equation (DPE, for short) for (3.4) is given as follows:

$$\begin{aligned} 0 &= U_t(y) + \sup_{\pi^1\in K_1}\inf_{\pi^0\in K_0}\sup_{\nu\in\mathbf{R}^d}\Big\{ r(y,\pi)\beta U_\beta(y) + (DU(y)-\pi^0)^\top diag[s]\sigma(y,\pi)^\top\nu \\ &\qquad\qquad + \frac{1}{2}\mathbf{Tr}\Big[\big\{diag[s]\sigma^\top\sigma(y,\pi)diag[s]\big\}D^2U(y)\Big]\Big\} \\ &= \begin{cases} \sup_{\pi^1\in K_1}\mathcal{G}^{DU(y),\pi^1}U(y)\,, & \text{if } DU(y)\in K_0, \\ \quad +\infty \qquad , & \text{if } DU(y)\notin K_0, \end{cases} \end{aligned} \tag{3.7}$$

for $y = (t,\beta,s) \in [0,T)\times(0,\infty)^{n+1}$, where $D\varphi$ and $D^2\varphi$ are the first and second order differentials of $\varphi$ with respect to the variable $s$ and

$$\mathcal{G}^\pi\varphi(y) = \varphi_t(y) + r(y,\pi)\beta\varphi_\beta(y) + \frac{1}{2}\mathbf{Tr}\Big[\big\{diag[s]\sigma^\top\sigma(y,\pi)diag[s]\big\}D^2\varphi(y)\Big].$$

Combining (3.7) with (3.1), we have the DPE

$$\min\Big\{-\sup_{\pi^1\in K_1}\mathcal{G}^{DU(y),\pi^1}U(y),\ \ \mathcal{H}_0(DU(y))\Big\}=0,\qquad y\in[0,T)\times(0,\infty)^{n+1}. \tag{3.8}$$

Similarly, the DPE derived from (3.5) is characterized as

$$\min\Big\{-\sup_{\pi^0\in K_0}\mathcal{G}^{\pi^0,DL(y)}L(y),\ \ \mathcal{H}_1(DL(y))\Big\}=0,\qquad y\in[0,T)\times(0,\infty)^{n+1},$$

where $L(y):=-\widetilde{L}(y)$.

We are now in the position to provide some conditions on the payoff function $g$ and convex set $K_0$.

**Assumption 3.1**

(i) *There are constants $c_0, l>0$ and $\gamma_0\in K_0$ such that*

$$g(\beta,s)\le c_0(\beta^l+\beta^{-l})+\gamma_0^\top s,\qquad (\beta,s)\in(0,\infty)\times\mathbf{R}^n_+. \tag{3.9}$$

(ii) *Either one of the following conditions holds:*

$$\bullet\ \ g \text{ is continuous},\qquad or\qquad \bullet\ \ \widehat{g}_0 \text{ is continuous and } \widehat{g}_0=\widehat{(g_*)_0}\ , \tag{3.10}$$

*where*

$$g_*(z):=\liminf_{\varepsilon\downarrow 0}\big\{g(z')\ :\ z'\in(0,\infty)\times\mathbf{R}^n_+ \text{ and } |z-z'|\le\varepsilon\big\},\qquad z\in\mathbf{R}^{n+1}_+.$$

(iii) *For any $q,q'\in\mathbf{R}^n$ satisfying $q'-q\in\mathbf{R}^n_+$ and $|q_k|=|q'_k|$, $k=1,\dots,n$, we have*

$$\delta_0(q)\ge\delta_0(q'). \tag{3.11}$$

**Example 3.2** Let us consider the following two examples.

(i) $K_0$ is the closed ball $\mathrm{B}_\rho(0)$ centered at 0 with radius $\rho>0$. Then $\delta_0(q)=\rho|q|$ satisfies (3.11).

(ii) (*Rectangular constraints*) $K_0=J_1\times\cdots\times J_n$ with $J_k=[-\eta_k,\xi_k]$, $0\le\xi_k\le\eta_k<\infty$. Then $\delta_0(q)=\sum_{k=1}^n(\xi_k q_k^+ +\eta_k q_k^-)$ satisfies (3.11).

The following theorem characterizes the value function $U$ as a viscosity solution of the DPE (3.8). For the notion and general theory of viscosity solutions, we recommend readers to refer to the User's Guide by Crandall *et al.*[5].

**Theorem 3.3** *Let* (3.2) *and* (3.9) *hold. Then $U$ satisfies the following expressions:*

(i) (Growth condition) *For all $(t,\beta,s)\in[0,T]\times(0,\infty)\times(0,\infty)^n$,*

$$0\le U(t,\beta,s)\le c_0(\beta^l+\beta^{-l})e^{l\|r\|_\infty T}+\gamma_0^\top s. \tag{3.12}$$

(ii) (Supersolution) *For any smooth test function $\varphi$ and local minimizer $y=(t,\beta,s)\in[0,T)\times\mathbf{R}_+^{n+1}$ of $(U_*-\varphi)$ on $[0,T]\times\mathbf{R}_+^{n+1}$, we have*

$$\min\Big\{-\sup_{\pi^1\in K_1}\mathcal{G}^{D\varphi(y),\pi^1}\varphi(y),\ \sup_{p\in\mathbf{R}^n}\mathcal{H}_0(D^{s,p}\varphi(y))\Big\}\geq 0. \tag{3.13}$$

(iii) (Subsolution) *For any smooth test function $\varphi$ and local maximizer $y=(t,\beta,s)\in[0,T)\times\mathbf{R}_+^{n+1}$ of $(U^*-\varphi)$ on $[0,T]\times\mathbf{R}_+^{n+1}$, we have*

$$\min\Big\{-\sup_{\pi^1\in K_1}\mathcal{G}^{D\varphi(y),\pi^1}\varphi(y),\ \widetilde{\mathcal{H}}_0(D\varphi(y):s)\Big\}\leq 0. \tag{3.14}$$

(iv) (Terminal condition) $\quad U_*(T,z)\geq\widehat{(g_*)}_0(z),\qquad z\in(0,\infty)^{n+1}$.
*Moreover if $\gamma_0$ in* (3.9) *is an element of* $int(K_0\cap\mathbf{R}_+^n)$ *and* (3.10)-(3.11) *are satisfied, then* $\quad U_*(T,z)=U^*(T,z)=\widehat{g}_0(z),\qquad z\in(0,\infty)^{n+1}$.

*Here the upper (resp. lower) semicontinuous envelope $U^*$ (resp. $U_*:=-(-U)^*$) of $U$ is defined as*

$$U^*(y):=\limsup_{\varepsilon\downarrow 0}\big\{U(y')\ :\ |y-y'|\leq\varepsilon,\ \ y'\in[0,T)\times(0,\infty)^{n+1}\big\},\quad y\in[0,T]\times\mathbf{R}_+^{n+1},$$
$$D^{s,p}\varphi:=(D_1^{s,p}\varphi,\ldots,D_n^{s,p}\varphi)^\top\qquad \textit{with}\qquad D_j^{s,p}\varphi:=D_{s_j}\varphi\mathbb{1}_{\{s_j>0\}}+p_j\mathbb{1}_{\{s_j=0\}},$$

*and* $\widetilde{\mathcal{H}}_0(p:s):=\mathcal{H}_0(p)\mathbb{1}_{\{s\in(0,\infty)^n\}}+\infty\mathbb{1}_{\{s\in\partial\mathbf{R}_+^n\}}$.

*Proof* Here we prove only (3.12). The other claims are proved in §5.

It is clear that $U(t,\beta,s)\geq 0$. Let $\pi^0(\cdot)\equiv\gamma_0$ and $y=(t,\beta,s)\in[0,T)\times(0,\infty)^{n+1}$ be arbitrary. For any $\pi^1\in\Pi^1$ and $\nu\in\mathcal{D}$, we then have

$$\begin{aligned}
&\mathbb{E}^y\left[g(B^{\pi,\nu}(T),S^{\pi,\nu}(T))-\int_t^T\pi^0(u)^\top diag[S^{\pi,\nu}(u)]\sigma(a(u))^\top\nu(u)du\right]\\
&\leq c_0\mathbb{E}^y\left[B^{\pi,\nu}(T)^l+B^{\pi,\nu}(T)^{-l}\right]+\mathbb{E}^y\left[\gamma_0^\top S^{\pi,\nu}(T)-\int_t^T\gamma_0^\top diag[S^{\pi,\nu}(u)]\sigma(a(u))^\top\nu(u)du\right]\\
&\leq c_0(\beta^l+\beta^{-l})e^{l\|r\|_\infty T}+\gamma_0^\top s,
\end{aligned}$$

this provides the second inequality in (3.12). $\square$

Finally we consider a verification theorem for the DPE (3.8).

**Corollary 3.4** (Verification Theorem) *Assume* (3.2) *and* $g\leq\widehat{(g_*)}_0$. *Let* $u\in C([0,T]\times(0,\infty)^{n+1})\cap C^{1,1,2}([0,T)\times(0,\infty)\times(0,\infty)^n)$ *be solution of*

$$\begin{aligned}
\sup_{\pi^1\in K_1}\mathcal{G}^{Du(y),\pi^1}u(y)&=0, && y\in[0,T)\times(0,\infty)^{n+1},\\
Du(y)&\in K_0, && y\in[0,T)\times(0,\infty)^{n+1},\\
u(T,z)&=\widehat{(g_*)}_0(z), && z\in(0,\infty)^{n+1},
\end{aligned}$$

$$u(t,z) \le c_0\Big(1 + |z|^l + \prod_{j=0}^{n} z_j^{-l}\Big), \qquad\qquad z \in (0,\infty)^{n+1},$$

*where* $c_0,\ l > 0$ *are constants. Then* $u = U$ *on* $[0,T) \times (0,\infty)^{n+1}$*.*

<u>*Proof*</u> **1.** First, we show $u \ge U$. Fix arbitrary $y = (t,z) \in [0,T) \times (0,\infty)^{n+1}$. Let $\pi^1 \in \Pi^1$ and $\nu \in \mathcal{D}$ be arbitrary, and define $\pi^0(v) := Du(Y_y^{\pi,\nu}(v)),\ v \in [t,T)$. We notice that the SDE (3.6) has a unique solution since $r, b$ and $\sigma$ are Lipschitz functions in the $(z,\pi)$ variable, uniformly in $t$. Applying Itô's lemma to $u$, we see that

$$\begin{aligned}
&\mathbb{E}^y\Big[\widehat{(g_*)}_0(B^{\pi,\nu}(T), S^{\pi,\nu}(T)) - \int_t^T \pi^0(v)^\top diag[S^{\pi,\nu}(v)]\sigma(a(v))^\top \nu(v)dv\Big] \\
&\quad = u(y) + \mathbb{E}^y\Big[\int_t^T \mathcal{G}^\pi u(Y^{\pi,\nu}(v))dv\Big] \qquad\qquad\qquad\qquad\qquad \le u(y).
\end{aligned}$$

Hence we have $u(y) \ge U(y)$. (This inequality guarantees that $U$ is locally bounded and satisfies (3.13) without the condition (3.9).)
**2.** To prove the reverse inequality, we suppose that $2\zeta := u(t_*, z_*) - U_*(t_*, z_*) > 0$ for some $(t_*, z_*) \in [0,T) \times (0,\infty)^{n+1}$, and let us work towards a contradiction. Define

$$\begin{aligned}
v(t,z) &:= \frac{U_*(t,z)}{\xi(z)} = \frac{U_*(t,z)}{1 + |z|^{l+1} + \prod_{j=0}^{n} z_j^{-(l+1)}}, \\
\varphi(t,z) &:= \frac{u(t,z)}{\xi(z)} - \frac{\zeta}{\xi(z)}\,\frac{t-T}{t_* - T},
\end{aligned}$$

and $\psi := v - \varphi$. Since

$$\psi(t_*, z_*) = -\frac{\zeta}{\xi(z_*)} < 0, \qquad \psi(T,z) = \lim_{|z|\to\infty} \psi(t,z) = \lim_{z \to \tilde{z} \in \partial\mathbf{R}_+^{n+1}} \psi(t,z) = 0,$$

$\psi$ achieves its minimum at some $(\bar{t}, \bar{z}) \in [0,T) \times (0,\infty)^{n+1}$. Hence (3.13) implies that $-\sup_{\pi^1 \in K_1} \mathcal{L}^{\pi^1}\varphi(\bar{t},\bar{z}) \ge 0$, where

$$\begin{aligned}
\mathcal{L}^{\pi^1}\varphi(t,z) :=\ & \xi(z)\mathcal{G}^{D(\xi\varphi)(t,z),\pi^1}\varphi(t,z) + \varphi(t,z)\mathcal{G}^{D(\xi\varphi)(t,z),\pi^1}\xi(z) \\
& + \frac{1}{2}\mathbf{Tr}\Big[\Sigma(t,z,D(\xi\varphi)(t,z),\pi^1)\big\{D\varphi(t,z)D\xi(z)^\top + D\xi(z)D\varphi(t,z)^\top\big\}\Big]
\end{aligned}$$

and $\Sigma(t,\beta,s,\pi) := diag[s]\sigma^\top\sigma(t,\beta,s,\pi)diag[s]$. However, this contradicts with

$$-\sup_{\pi^1 \in K_1} \mathcal{L}^{\pi^1}\varphi(\bar{t},\bar{z}) = -\sup_{\pi^1 \in K_1} \mathcal{G}^{Du(\bar{t},\bar{z}),\pi^1} u(\bar{t},\bar{z}) - \frac{\zeta}{T - t_*} = -\frac{\zeta}{T - t_*} < 0. \qquad \square$$

Needless to say, it is easy to deduce the analogous results on the lower hedging cost $-L$. Hence we avoid going into details here.

# 4 Proof of Theorem 2.1

To prove Theorem 2.1, we need the following lemma:

**Lemma 4.1 (Cvitanić *et al.*[8])** *Let $K \subset \mathbf{R}^d$ be a compact, convex set which contains the origin, and $\delta$ the support function $\delta(z) := \sup_{y\in K}(y^\top z)$, $z \in \mathbf{R}^d$. For an $\mathcal{F}_T$-measurable random variable $\eta$, we define the process $Y$ as*

$$Y(t) := \operatorname*{ess\,sup}_{\nu\in\mathcal{D}} \mathbb{E}_\nu \left[ \eta - \int_t^T \delta(\nu(u))du \middle| \mathcal{F}_t \right], \qquad 0 \le t \le T. \tag{4.1}$$

*If $\mathbb{E}_\nu[\sup_{0\le t\le T}|Y(t)|] < \infty$ for every $\nu \in \mathcal{D}$, then there are a $K$-valued, $\mathbb{F}$-progressively measurable process $\alpha$ and a predictable increasing, right-continuous process $A$ with $A(0) = 0$ such that*

$$Y(t) = \eta - \int_t^T \alpha(u)^\top dW(u) + A(T) - A(t) \qquad a.s., \qquad 0 \le t \le T. \tag{4.2}$$

**Remark 4.2** Cvitanić *et al.*[8] used this result to find a minimal solution of a backward SDE with constraints. Also, Sekine[21] used the analogous result to obtain a characterization of the upper hedging cost under delta constraints in terms of an associated stochastic control problem.

*Proof of Theorem 2.1:* Denote by $\widehat{h}_{up}$ (resp. $\widehat{h}_{low}$) the right-hand side of (2.1) (resp. (2.2)). Let $\varepsilon, \rho, l, m > 0$ be arbitrary. Let $K$ be the closed ball $\mathrm{B}_\varepsilon(0) = \{|z| \le \varepsilon\} \subset \mathbf{R}^d$.
**1.** We first prove $h_{up} \ge \widehat{h}_{up}$. If the set of (1.2) is empty, then $h_{up} = \infty \ge \widehat{h}_{up}$. Suppose that the set of (1.2) contains an element $x_0$. Then, for all $\pi^1 \in \Pi^1$ there exists $\pi^0 \in \Pi^0$ such that

$$Z_\nu(\tau)x_0 \ge Z_\nu(\tau)\left(C^\pi(\tau) - X_0^{0,\pi}(\tau)\right)^+ \quad \text{a.s.}, \qquad \tau \in \mathcal{T}^\pi, \quad \nu \in \mathcal{D}.$$

Hence we get

$$x_0 \ge \inf_{\pi^0\in\Pi^0} \sup_{\tau\in\mathcal{T}^\pi} \sup_{\nu\in\mathcal{D}} \mathbb{E}_\nu \left[ \left(C^\pi(\tau) - X_0^{0,\pi}(\tau)\right)^+ \right], \qquad \pi^1 \in \Pi^1,$$

which implies $h_{up} \ge \widehat{h}_{up}$.
**2.** Next we show $h_{low} \le \widehat{h}_{low}$. Since $0 \in \Pi^1$, the set of (1.3) contains the origin and thus $h_{low} \ge 0$. Let $x_1$ be arbitrary element of the set of (1.3). Then, for any $\pi^0 \in \Pi^0$ there exist $\pi^1 \in \Pi^1$ and $\tau \in \mathcal{T}^\pi$ such that

$$0 \le Z_\nu(\tau)(x_1 \wedge m) \le Z_\nu(\tau)\left\{\left(C^\pi(\tau) + X_1^{0,\pi}(\tau)\right) \wedge m\right\} \quad \text{a.s.}, \qquad \nu \in \mathcal{D}.$$

Thus we obtain

$$x_1 \wedge m \le \sup_{\pi^1\in\Pi^1} \sup_{\tau\in\mathcal{T}^\pi} \inf_{\nu\in\mathcal{D}} \mathbb{E}_\nu \left[ \left(C^\pi(\tau) + X_1^{0,\pi}(\tau)\right) \wedge m \right], \qquad \pi^0 \in \Pi^0,$$

which yields $h_{low} \le \widehat{h}_{low}$.

**3.** Let $\pi \in \Pi$, $\tau \in \mathcal{T}^\pi$ be arbitrary. To prove $h_{up} \leq \widehat{h}_{up}$, define the process $Y$ as (4.1) with the $\mathcal{F}_T$-measurable random variable $\eta := \big(C^\pi(\tau) - X_0^{0,\pi}(\tau)\big)^+ \wedge m$. Since the support function $\delta$ is non-negative, by means of (4.2), we have

$$\begin{aligned} \eta &= Y(0) + \int_0^T \alpha_\varepsilon(t)^\top dW(t) - A(T) \\ &\leq \sup_{\nu \in \mathcal{D}} \mathbb{E}_\nu \left[ \big(C^\pi(\tau) - X_0^{0,\pi}(\tau)\big)^+ \right] + \int_0^T \alpha_\varepsilon(t)^\top dW(t) \qquad \text{a.s.} \end{aligned} \tag{4.3}$$

and $\mathbb{E}\left[\left(\int_0^T \alpha_\varepsilon(t)^\top dW(t)\right)^2\right] \leq T\varepsilon^2$. By possibly passing to a subsequence $\{\alpha_{\varepsilon'}\}$ and letting $\varepsilon' \downarrow 0$, $m \to \infty$, we get

$$C^\pi(\tau) - X_0^{0,\pi}(\tau) \leq \sup_{\nu \in \mathcal{D}} \mathbb{E}_\nu \left[ \big(C^\pi(\tau) - X_0^{0,\pi}(\tau)\big)^+ \right] \qquad \text{a.s.} \tag{4.4}$$

For all $\pi^1 \in \Pi^1$, we can choose $\pi^0_\rho \in \Pi^0$ such that

$$\sup_{\tau \in \mathcal{T}^{\pi_\rho}} \sup_{\nu \in \mathcal{D}} \mathbb{E}_\nu \left[ \big(C^{\pi_\rho}(\tau) - X_0^{0,\pi_\rho}(\tau)\big)^+ \right] \leq \inf_{\pi^0 \in \Pi^0} \sup_{\tau \in \mathcal{T}^\pi} \sup_{\nu \in \mathcal{D}} \mathbb{E}_\nu \left[ \big(C^\pi(\tau) - X_0^{0,\pi}(\tau)\big)^+ \right] + \rho,$$

where $\pi_\rho = (\pi^0_\rho, \pi^1)$. Hence we have

$$C^{\pi_\rho}(\tau) - X_0^{0,\pi_\rho}(\tau) \leq \widehat{h}_{up} + \rho \qquad \text{a.s.,} \qquad \tau \in \mathcal{T}^{\pi_\rho},$$

which means $h_{up} \leq \widehat{h}_{up} + \rho$, and thus we obtain $h_{up} \leq \widehat{h}_{up}$ by $\rho \downarrow 0$.
**4.** To show $h_{low} \geq \widehat{h}_{low}$, let $\pi \in \Pi$, $\tau \in \mathcal{T}^\pi$ be arbitrary. Define the process $Y$ as (4.1) with $-\eta := \big(C^\pi(\tau) + X_1^{0,\pi}(\tau)\big) \wedge m \vee (-l)$. Then, by the similar arguments to (4.3)-(4.4), we get

$$\begin{aligned} C^\pi(\tau) + X_1^{0,\pi}(\tau) &= \lim_{l \to \infty} \big(C^\pi(\tau) + X_1^{0,\pi}(\tau)\big) \vee (-l) \\ &\geq \inf_{\nu \in \mathcal{D}} \mathbb{E}_\nu \left[ \big(C^\pi(\tau) + X_1^{0,\pi}(\tau)\big) \wedge m \right] \qquad \text{a.s.} \end{aligned} \tag{4.5}$$

Fix arbitrary $\pi^0 \in \Pi^0$. Then, for certain $\widehat{\pi}^1_\rho \in \Pi^1$ and $\widehat{\tau}_\rho \in \mathcal{T}^{\widehat{\pi}_\rho}$ we have

$$\inf_{\nu \in \mathcal{D}} \mathbb{E}_\nu \left[ \Big(C^{\widehat{\pi}_\rho}(\widehat{\tau}_\rho) + X_1^{0,\widehat{\pi}_\rho}(\widehat{\tau}_\rho)\Big) \wedge m \right] + \rho \geq \sup_{\pi^1 \in \Pi^1} \sup_{\tau \in \mathcal{T}^\pi} \inf_{\nu \in \mathcal{D}} \mathbb{E}_\nu \left[ \big(C^\pi(\tau) + X_1^{0,\pi}(\tau)\big) \wedge m \right],$$

where $\widehat{\pi}_\rho = (\pi^0, \widehat{\pi}^1_\rho)$. Therefore we get

$$C^{\widehat{\pi}_\rho}(\widehat{\tau}_\rho) + X_1^{0,\widehat{\pi}_\rho}(\widehat{\tau}_\rho) + \rho \geq \inf_{\pi^0 \in \Pi^0} \sup_{\pi^1 \in \Pi^1} \sup_{\tau \in \mathcal{T}^\pi} \inf_{\nu \in \mathcal{D}} \mathbb{E}_\nu \left[ \big(C^\pi(\tau) + X_1^{0,\pi}(\tau)\big) \wedge m \right] \qquad \text{a.s.}$$

Denote by $h_m$ the right-hand side of the above inequality, and set

$$\widetilde{\pi}_\rho := \big(\pi^0, \widehat{\pi}^1_\rho \mathbb{1}_{\{h_m \geq \rho\}}\big) \in \Pi, \qquad \widetilde{\tau}_\rho := \widehat{\tau}_\rho \mathbb{1}_{\{h_m \geq \rho\}} + T \mathbb{1}_{\{h_m < \rho\}} \in \mathcal{T}^{\widetilde{\pi}_\rho}.$$

Then we obtain

$$C^{\widetilde{\pi}_\rho}(\widetilde{\tau}_\rho) + X_1^{0,\widetilde{\pi}_\rho}(\widetilde{\tau}_\rho) \geq (h_m - \rho)^+ \quad \text{a.s.}$$

which implies that $h_m - \rho \leq h_{low}$. Letting $\rho \downarrow 0$ and $m \to \infty$, we have $\widehat{h}_{low} \leq h_{low}$.
**5.** Next we prove (2.3). (2.1) says that $h_{up}$ is not less than the right-hand side of (2.3). To prove the reverse inequality, let $\pi \in \Pi$, $\tau \in \mathcal{T}^\pi$ and assume $\mathbb{E}[|X_0^{0,\pi}(\tau)|^p] < \infty$ for some $p > 1$. Define the process $Y$ as (4.1) with $\eta := (C^\pi(\tau) - X_0^{0,\pi}(\tau)) \wedge m$. Since

$$-\mathbb{E}\left[|X_0^{0,\pi}(\tau)|\,\middle|\,\mathcal{F}_t\right] \leq \mathbb{E}[\eta|\mathcal{F}_t] \leq Y(t) \leq m \qquad \text{a.s.},$$

the Hölder inequality and Doob's maximal inequality show

$$\begin{aligned}
\mathbb{E}_\nu\left[\sup_{0\leq t\leq T}|Y(t)|\right] &\leq m + \mathbb{E}_\nu\left[\sup_{0\leq t\leq T}\mathbb{E}\left[|X_0^{0,\pi}(\tau)|\,\middle|\,\mathcal{F}_t\right]\right] \\
&\leq m + \mathbb{E}\left[Z_\nu(T)^q\right]^{\frac{1}{q}}\mathbb{E}\left[\sup_{0\leq t\leq T}\mathbb{E}\left[|X_0^{0,\pi}(\tau)|\,\middle|\,\mathcal{F}_t\right]^p\right]^{\frac{1}{p}} \\
&\leq m + q\,\mathbb{E}\left[Z_\nu(T)^q\right]^{\frac{1}{q}}\mathbb{E}\left[|X_0^{0,\pi}(\tau)|^p\right]^{\frac{1}{p}} \qquad < \infty
\end{aligned} \tag{4.6}$$

for any $\nu \in \mathcal{D}$ and $q = p/(p-1)$. By the similar arguments to (4.3)-(4.4), hence, we have

$$C^\pi(\tau) - X_0^{0,\pi}(\tau) \leq \sup_{\nu\in\mathcal{D}}\mathbb{E}_\nu\left[C^\pi(\tau) - X_0^{0,\pi}(\tau)\right] \qquad \text{a.s.}$$

We also note that for all $\pi^1 \in \Pi^1$ there exists $\pi_\rho^0 \in \Pi^0$ such that

$$\left(\sup_{\tau\in\mathcal{T}^{\pi_\rho}}\sup_{\nu\in\mathcal{D}}\mathbb{E}_\nu\left[C^{\pi_\rho}(\tau) - X_0^{0,\pi_\rho}(\tau)\right]\right)^+ \leq \left(\inf_{\pi^0\in\Pi^0}\sup_{\tau\in\mathcal{T}^\pi}\sup_{\nu\in\mathcal{D}}\mathbb{E}_\nu\left[C^\pi(\tau) - X_0^{0,\pi}(\tau)\right]\right)^+ + \rho,$$

where $\pi_\rho = (\pi_\rho^0, \pi^1)$. From two inequalities above, we know that $h_{up} - \rho$ is not greater than the right-hand side of (2.3). Since $\rho > 0$ is arbitrary, we have (2.3).
**6.** Finally we show (2.4). Denote by $\widetilde{h}_{low}$ the right-hand side of (2.4). It is clear $h_{low} \leq \widetilde{h}_{low}$ by (2.2). Let $\pi \in \Pi$, $\tau \in \mathcal{T}^\pi$, and assume $\mathbb{E}[C^\pi(\tau)^p + |X_0^{0,\pi}(\tau)|^p] < \infty$ for some $p > 1$. Define the process $Y$ as (4.1) with $-\eta := (C^\pi(\tau) + X_1^{0,\pi}(\tau)) \vee (-l)$. Clearly,

$$-l \leq -Y(t) \leq \mathbb{E}[-\eta|\mathcal{F}_t] \leq \mathbb{E}\left[C^\pi(\tau) + \left|X_1^{0,\pi}(\tau)\right|\,\middle|\,\mathcal{F}_t\right] \quad \text{a.s.}$$

Therefore the same calculus as (4.6) gives

$$\mathbb{E}_\nu\left[\sup_{0\leq t\leq T}|Y(t)|\right] \leq l + q\,\mathbb{E}\left[Z_\nu(T)^q\right]^{\frac{1}{q}}\mathbb{E}\left[\left(C^\pi(\tau) + |X_1^{0,\pi}(\tau)|\right)^p\right]^{\frac{1}{p}} < \infty$$

for any $\nu \in \mathcal{D}$ and $q = p/(p-1)$. Thus, by the similar arguments to (4.5), we have

$$C^\pi(\tau) + X_1^{0,\pi}(\tau) \geq \inf_{\nu\in\mathcal{D}}\mathbb{E}_\nu\left[C^\pi(\tau) + X_1^{0,\pi}(\tau)\right] \qquad \text{a.s.}$$

Fix arbitrary $\pi^0 \in \Pi^0$. Then we can choose $\widehat{\pi}^1_\rho \in \Pi^1$ and $\widehat{\tau}_\rho \in \mathcal{T}^{\widehat{\pi}_\rho}$ such that

$$\left(\inf_{\nu\in\mathcal{D}} \mathbb{E}_\nu \left[C^{\widehat{\pi}_\rho}(\widehat{\tau}_\rho) + X_1^{0,\widehat{\pi}_\rho}(\widehat{\tau}_\rho)\right]\right) \wedge m + \rho$$
$$\geq \left(\sup_{\pi^1\in\Pi^1} \sup_{\tau\in\mathcal{T}^\pi} \inf_{\nu\in\mathcal{D}} \mathbb{E}_\nu \left[C^\pi(\tau) + X_1^{0,\pi}(\tau)\right]\right) \wedge m \quad \geq \widetilde{h}_{low} \wedge m,$$

where $\widehat{\pi}_\rho = (\pi^0, \widehat{\pi}^1_\rho)$. Therefore we can show $\widetilde{h}_{low} \wedge m - \rho \leq h_{low}$ along the same line as Step 4. Letting $\rho \downarrow 0$ and $m \to \infty$, we obtain (2.4). Hence the proof is complete. □

# 5 Proof of Theorem 3.3

We shall denote $Y^{\pi,\nu}_y(u) := (u, B^\pi_{t,\beta}(u), S^{\pi,\nu}_{t,s}(u))$, $u \in [t,T]$ for each $\pi \in \Pi$, $\nu \in \mathcal{D}$ and $y = (t, \beta, s) \in [0,T] \times \mathbf{R}^{n+1}_+$. Throughout this section, we assume (3.2) and (3.9).

## 5.1 Dynamic programming principle

We first establish the dynamic programming principle (DPP, for short): For every $y = (t, \beta, s) \in [0,T] \times (0,\infty)^{n+1}$ and $\mathbb{F}$-stopping time $\tau$ taking values in $[t,T]$ a.s.,

$$U(y) = \sup_{\pi^1\in\Pi^1} \inf_{\pi^0\in\Pi^0} \sup_{\nu\in\mathcal{D}} \mathbb{E}^y \left[U(Y^{\pi,\nu}(\tau)) - \int_t^\tau \pi^0(u)^\top diag[S^{\pi,\nu}(u)]\sigma(Y^{\pi,\nu}(u), \pi(u))^\top \nu(u) du\right]. \tag{5.1}$$

By virtue of the arguments in §2 and §4, we know that this principle is equivalent to the following lemma:

**Lemma 5.1 (Soner & Touzi[22])** *For every $y = (t, \beta, s) \in [0,T] \times (0,\infty)^{n+1}$ and $\mathbb{F}$-stopping time $\tau$ satisfying $t \leq \tau \leq T$ a.s., we have*

$$U(y) = \inf\left\{x_0 \geq 0 \;\middle|\; \begin{array}{l} \forall \pi^1 \in \Pi^1, \;\; \exists \pi^0 \in \Pi^0 \quad s.t. \\ \quad X^{t,x_0,\pi}_{0,y}(\tau) \geq U(Y^{\pi,\nu}_y(\tau)) \quad a.s. \end{array}\right\},$$

*where*

$$X^{t,x_0,\pi}_{0,y}(u) = x_0 + \int_t^u \pi^0(q)^\top dS^{\pi,\nu}_y(q), \qquad t \leq u \leq T.$$

*Proof* It immediately follows from the arguments in §3 of Soner & Touzi[22]. □

## 5.2 Supersolution property

**Lemma 5.2** *$U_*$ satisfies* (3.13).

*Proof* **1.** Let $y = (t, z) \in [0, T) \times \mathbf{R}_+^{n+1}$ and $\varphi$ be an $\mathbf{R}$-valued smooth test function on $[0, T] \times \mathbf{R}_+^{n+1}$, and suppose that

$$0 = (U_* - \varphi)(y) = \min_{[0,T]\times\mathbf{R}_+^{n+1}} (U_* - \varphi).$$

Let $\{y_m\}_{m\geq 1} \subset [0, T) \times (0, \infty)^{n+1}$ be a sequence satisfying

$$y_m = (t_m, z_m) \to y \qquad \text{and} \qquad U(y_m) \to U_*(y) \qquad \text{as } m \to \infty.$$

Set

$$\varepsilon_m^2 := U(y_m) - \varphi(y_m) + m^{-1} \quad \to 0 \qquad \text{as } m \to \infty.$$

Fix arbitrary $\pi^1(\cdot) \equiv \pi^1 \in K_1$ and $\nu(\cdot) \equiv \nu \in \mathbf{R}^d$. For each $\pi^0 \in \Pi^0$ and sufficiently large number $m$, define

$$\tau_m := (t_m + \varepsilon_m) \wedge \inf\left\{u \geq t_m \ : \ \left|H_{y_m}^{\pi,\nu}(u) - H_{y_m}^{\pi,\nu}(t_m)\right| \geq 1\right\} < T,$$

where $H_{y_m}^{\pi,\nu}(u) := \log B_{y_m}^{\pi}(u) + \sum_{j=1}^n \log S_{y_m,j}^{\pi,\nu}(u)$.

Since $\varphi \leq U_* \leq U$ on $[0, T) \times (0, \infty)^{n+1}$, it follows from the DPP (5.1) that there is a $\pi_m^0 \in \Pi^0$ such that

$$\begin{aligned}
\varepsilon_m^2 &\geq -\varphi(y_m) + \frac{1}{m} + \inf_{\pi^0\in\Pi^0} \mathbb{E}^{y_m}\left[U(Y^{\pi,\nu}(\tau_m)) - \int_{t_m}^{\tau_m} \pi^0(u)^\top diag[S^{\pi,\nu}(u)]\sigma(a(u))^\top \nu du\right] \\
&\geq -\varphi(y_m) + \mathbb{E}^{y_m}\left[U(Y^{\pi_m^0,\pi^1,\nu}(\tau_m)) - \int_{t_m}^{\tau_m} \pi_m^0(u)^\top diag[S^{\pi_m^0,\pi^1,\nu}(u)]\sigma(a_m(u))^\top \nu du\right] \\
&\geq -\varphi(y_m) + \mathbb{E}^{y_m}\left[\varphi(Y^{\pi_m^0,\pi^1,\nu}(\tau_m)) - \int_{t_m}^{\tau_m} \pi_m^0(u)^\top diag[S^{\pi_m^0,\pi^1,\nu}(u)]\sigma(a_m(u))^\top \nu du\right] \\
&= \mathbb{E}^{y_m}\Bigg[\int_{t_m}^{\tau_m} \Big\{\left(D\varphi(Y^{\pi_m^0,\pi^1,\nu}(u)) - \pi_m^0(u)\right)^\top diag[S^{\pi_m^0,\pi^1,\nu}(u)]\sigma(a_m(u))^\top \nu \\
&\qquad\qquad + \mathcal{G}^{\pi_m^0(u),\pi^1}\varphi(Y^{\pi_m^0,\pi^1,\nu}(u))\Big\} du\Bigg] \\
&=: \mathbb{E}^{y_m}\left[\int_{t_m}^{\tau_m} F^{\pi^1,\nu}(Y^{\pi_m^0,\pi^1,\nu}(u), \pi_m^0(u)) du\right] \\
&\geq -\int_{t_m}^{t_m+\varepsilon_m} \mathbb{E}^{y_m}\left[\left|F^{\pi^1,\nu}(Y^{\pi_m^0,\pi^1,\nu}(u), \pi_m^0(u)) - F^{\pi^1,\nu}(y, \pi_m^0(u))\right| \mathbb{1}_{\{u\leq\tau_m\}}\right] du \\
&\quad + \mathbb{E}^{y_m}[\tau_m - t_m] \inf_{\pi^0\in K_0} F^{\pi^1,\nu}(y, \pi^0) \\
&\geq -C'\varepsilon_m(|y|\varepsilon_m^{1/2} + |y - y_m|) \\
&\quad + \left(\varepsilon_m \mathbb{P}\{\tau_m = t_m + \varepsilon_m\} + \mathbb{E}^{y_m}[(\tau_m - t_m)\mathbb{1}_{\{\tau_m < t_m+\varepsilon_m\}}]\right) \inf_{\pi^0\in K_0} F^{\pi^1,\nu}(y, \pi^0),
\end{aligned}$$

where $a_m(u) = (Y^{\pi_m^0,\pi^1,\nu}(u), \pi_m^0(u), \pi^1)$, a constant $C' > 0$ is independent of $m$, and the last inequality follows from the standard results about solutions of SDEs with random coefficients. Dividing the above inequality by $\varepsilon_m$, and sending $m$ to infinity using the estimate

$$
\begin{aligned}
0 \le \varepsilon_m^{-1}\mathbb{E}^{y_m}[(\tau_m - t_m)\mathbb{1}_{\{\tau_m<t_m+\varepsilon_m\}}] \qquad & \le \mathbb{P}\{\tau_m < t_m + \varepsilon_m\} \\
\le \mathbb{P}\left\{\sup_{0\le u<\varepsilon_m} \left|H^{\pi,\nu}_{y_m}(t_m+u) - H^{\pi,\nu}_{y_m}(t_m)\right| \ge 1\right\} & \\
\le \mathbb{E}\left[\sup_{0\le u<\varepsilon_m} \left|H^{\pi,\nu}_{y_m}(t_m+u) - H^{\pi,\nu}_{y_m}(t_m)\right|^2\right] & \qquad\qquad \le C'\varepsilon_m,
\end{aligned}
$$

we have $\quad 0 \ge \inf_{\pi^0\in K_0} F^{\pi^1,\nu}(y,\pi^0)$ , and hence

$$
\begin{aligned}
0 &\ge \sup_{\pi^1\in K_1}\sup_{\nu\in\mathbf{R}^d}\inf_{\pi^0\in K_0} F^{\pi^1,\nu}(y,\pi^0) \\
&= \begin{cases} \sup_{\pi^1\in K_1} \mathcal{G}^{D\varphi(y),\pi^1}\varphi(y) , & \text{if } \inf_{\pi^0\in K_0} |(D\varphi(y)-\pi^0)diag[s]| = 0, \\ +\infty , & \text{otherwise.} \end{cases}
\end{aligned}
$$

Therefore we obtain (3.13). (For the sake of simplicity, we assume that $y$ is a global minimizer of $U_* - \varphi$. It easily see that we have (3.13) for any local minimizer $y$ along the same line.) □

### 5.3 Subsolution property

Let us introduce

$$
\mathcal{M}_0(\varphi) := \left\{ y \in [0,T]\times\mathbf{R}^{n+1}_+ \;:\; \min\left\{ -\sup_{\pi^1\in K_1}\mathcal{G}^{D\varphi(y),\pi^1}\varphi(y),\; \widetilde{\mathcal{H}}_0(D\varphi(y):s)\right\} > 0 \right\}
$$

for each smooth test function $\varphi$. Then we have the analogous result with Lemma 4.2 of Soner & Touzi[23].

**Lemma 5.3** *Let $J = (t_1,t_2) \subset [0,T]$, $\mathrm{B}_\rho(z_0)$ be the closed ball centered at $z_0 \in \mathbf{R}^{n+1}_+$ with radius $\rho > 0$, $A_\rho := \mathrm{B}_\rho(z_0)\cap\mathbf{R}^{n+1}_+$, and $\varphi$ be a smooth test function. If*

$$
J \times int\, A_{2\rho} \subset \mathcal{M}_0(\varphi), \tag{5.2}
$$

*then*

$$
\sup\left\{U^* - \varphi \;:\; J^0\times A_\rho\right\} \le \sup\left\{U - \varphi \;:\; \partial_p(J\times A_{2\rho})\right\}, \tag{5.3}
$$

*where $\partial_p(J\times A) := (t_1,t_2]\times\{\partial A\cap(0,\infty)^{n+1}\}\cup\{t_2\}\times int\, A$ and $J^0 := J\cup(\{t_1\}\cap\{0\})\cup(\{t_2\}\cap\{T\})$.*

*Proof* Let $y \in J^0\times A_\rho$ and $\{y_m\}_m \subset [t_1,t_2)\times int\, A_{2\rho}$ be a sequence satisfying

$$
y_m = (t'_m,\beta_m,s_m) \to y \qquad \text{and} \qquad U(y_m) \to U^*(y) \qquad \text{as } m\to\infty.
$$

Fix arbitrary $m \ge 1$. For any $\pi^1 \in \mathit{\Pi}^1$ and $\nu \in \mathcal{D}$, we define $\pi^0(u) := D\varphi(Y^{\pi,\nu}_{y_m}(u))$, $u \in [t'_m,T]$. Since $r, b$ and $\sigma$ are Lipschitz functions in the $(\beta,s,\pi)$ variable, uniformly in $t$, we notice that the SDE (3.6) has a unique solution and $(B^{\pi,\nu}_{y_m}, S^{\pi,\nu}_{y_m}) \in (0,\infty)^{n+1}$ a.e.

Define $\pi_*(u) := \pi(u \wedge \tau)$, $u \in [t'_m, T]$ with

$$\tau := \inf\left\{u > t'_m \ : \ (B^{\pi,\nu}_{y_m}(u), S^{\pi,\nu}_{y_m}(u)) \notin int A_{2\rho}\right\} \wedge t_2.$$

By means of (5.2), we see $\pi_*(u) \in \varPi$. Taking account of (5.1) and (5.2), we have

$$\begin{aligned}
U(y_m) &\le \sup_{\pi^1_* \in \varPi^1} \sup_{\nu \in \mathcal{D}} \mathbb{E}^{y_m}\left[U(Y^{\pi_*,\nu}(\tau)) - \int_{t'_m}^{\tau} \pi^0_*(u)^\top diag[S^{\pi_*,\nu}(u)]\sigma(Y^{\pi_*,\nu}(u), \pi_*(u))^\top \nu(u) du\right] \\
&\le \sup\left\{U - \varphi \ : \ \partial_p(J \times A_{2\rho})\right\} \\
&\quad + \sup_{\pi^1 \in \varPi^1} \sup_{\nu \in \mathcal{D}} \mathbb{E}^{y_m}\left[\varphi(Y^{\pi,\nu}(\tau)) - \int_{t'_m}^{\tau} \pi^0(u)^\top diag[S^{\pi,\nu}(u)]\sigma(Y^{\pi,\nu}(u), \pi(u))^\top \nu(u) du\right] \\
&\le \sup\left\{U - \varphi \ : \ \partial_p(J \times A_{2\rho})\right\} + \varphi(y_m) + \sup_{\pi^1 \in \varPi^1} \sup_{\nu \in \mathcal{D}} \mathbb{E}^{y_m}\left[\int_{t'_m}^{\tau} \mathcal{G}^\pi \varphi(Y^{\pi,\nu}(u)) du\right] \\
&\le \sup\left\{U - \varphi \ : \ \partial_p(J \times A_{2\rho})\right\} + \varphi(y_m).
\end{aligned}$$

Letting $m \to \infty$, we obtain $U^*(y) - \varphi(y) \le \sup\{U - \varphi \ : \ \partial_p(J \times A_{2\rho})\}$, and thus (5.3). $\square$

**Lemma 5.4** *$U$ satisfies* (3.14).

*Proof* Let $y = (t_0, z_0) \in [0,T) \times \mathbf{R}^{n+1}_+$ and $\varphi$ be an $\mathbf{R}$-valued smooth test function on $[0,T] \times \mathbf{R}^{n+1}_+$, and suppose that

$$(U^* - \varphi)(y) = (strict) \max_{[0,T]\times\mathbf{R}^{n+1}_+} (U^* - \varphi).$$

Suppose $y \in \mathcal{M}_0(\varphi)$. Then there is a $\rho \in (0, T - t_0)$ such that $J \times int A_{2\rho} \subset \mathcal{M}_0(\varphi)$, where $J := ((t_0 - \rho)^+, t_0 + \rho)$ and $A_\rho$ is as in Lemma 5.3. In view of Lemma 5.3, we get the contradiction:

$$(U^* - \varphi)(y) \le \sup_{J^0 \times A_\rho} (U^* - \varphi) \le \sup_{\partial_p(J\times A_{2\rho})} (U - \varphi) < (U^* - \varphi)(y).$$

Hence we know $y \in \mathcal{M}_0(\varphi)^c$, that is, (3.14) holds. $\square$

### 5.4 Terminal condition

**Lemma 5.5** $U_*(T, \cdot) \ge \widehat{(g_*)}_0$ *on* $(0,\infty)^{n+1}$.

*Proof* **1.** Fix arbitrary $t_0 \in [0,T)$ and $\beta_0 > 0$. First we prove that $U_*$ satisfies

$$\mathcal{H}_0(DU_*(t_0, \beta_0, s)) \ge 0, \qquad s \in (0,\infty)^n \qquad \text{in the viscosity sense.}$$

To this end, let $s_0 \in (0,\infty)^n$ and $\varphi$ be an $\mathbf{R}$-valued smooth test function on $\mathbf{R}^n_+$, and suppose that

$$0 = (U_*(t_0,\beta_0,\cdot) - \varphi)(s_0) = (strict) \min_{\mathbf{R}^n_+} (U_*(t_0,\beta_0,\cdot) - \varphi). \tag{5.4}$$

Let $\mathrm{B}_{2\rho} := \mathrm{B}_{2\rho}(\beta_0, s_0) \subset (0,\infty)^{n+1}$ be the closed ball centered at $(\beta_0, s_0)$ with radius $2\rho > 0$. Define

$$\varphi_m(t,\beta,s) := \varphi(s) - m[(t-t_0)^2 + (\beta-\beta_0)^2], \qquad M_m := \min_{[0,T]\times \mathrm{B}_{2\rho}} (U_* - \varphi_m)$$

for each $m \geq 1$. Since $U_*$ is lower semicontinuous, $M_m = (U_* - \varphi_m)(y_m)$ for some $y_m = (t_m, \beta_m, s_m) \in [0,T] \times \mathrm{B}_{2\rho}$. There exist then a $(t_*, \beta_*, s_*) \in [0,T] \times \mathrm{B}_{2\rho}$ and a relabeled subsequence $(t_m, \beta_m, s_m)$ such that $(t_m, \beta_m, s_m) \to (t_*, \beta_*, s_*)$ as $m \to \infty$.

Since $U_*$ is non-negative, we have

$$m[(t_m - t_0)^2 + (\beta_m - \beta_0)^2] - \varphi(s_m) \leq M_m \leq (U_* - \varphi_m)(t_0, \beta_0, s_0) = 0,$$

and thus $(t_m, \beta_m) \to (t_0, \beta_0)$ as $m \to \infty$. Further

$$\begin{aligned} \limsup_{m\to\infty} m[(t_m - t_0)^2 + (\beta_m - \beta_0)^2] &\leq \limsup_{m\to\infty} \{\varphi(s_m) - U_*(y_m)\} \\ &\leq \varphi(s_*) - U_*(t_0, \beta_0, s_*) \qquad \leq 0. \end{aligned}$$

In view of (5.4), this inequality provides $s_* = s_0$, $\lim_{m\to\infty} U_*(y_m) = U_*(t_0,\beta_0,s_0)$ and $(t_m, \beta_m, s_m) \in [0,T) \times \mathrm{B}_\rho$ for sufficiently large $m$. Hence, we know from Lemma 5.2 that

$$0 \leq \liminf_{m\to\infty} \mathcal{H}_0(D\varphi_m(t_m, \beta_m, s_m)) = \liminf_{m\to\infty} \mathcal{H}_0(D\varphi(s_m)) = \mathcal{H}_0(D\varphi(s_0)).$$

**2.** From the definition of $U_*$, we note

$$U_*(T,\beta,s) = \liminf_{\varepsilon\downarrow 0} \left\{ U_*(t_0,\beta_0,s_0) \ : \ \begin{array}{l} 0 < T - t_0 \leq \varepsilon, \\ |\beta - \beta_0| + |s - s_0| \leq \varepsilon, \ \ (\beta_0, s_0) \in \mathbf{R}^{n+1}_+. \end{array} \right\}$$

for $(\beta, s) \in \mathbf{R}^{n+1}_+$. Therefore it follows from the stability property that for each $\beta > 0$ $U_*(T,\beta,\cdot)$ satisfies

$$\mathcal{H}_0(DU_*(T,\beta,s)) \geq 0, \qquad s \in (0,\infty)^n \qquad \text{in the viscosity sense.}$$

**3.** Fix arbitrary $z \in \mathbf{R}^{n+1}_+$. Next we show $U_*(T,z) \geq g_*(z)$. Choose $\pi^1 \equiv 0$ and $\nu \equiv 0$. Let $\{y_m\}_{m\geq 1} \subset [0,T) \times (0,\infty)^{n+1}$ be a sequence satisfying

$$y_m = (t_m, \beta_m, s_m) \to y := (T, z) \qquad \text{and} \qquad U(y_m) \to U_*(y) \qquad \text{as } m \to \infty.$$

Then we have

$$U(y_m) + \frac{1}{m} \geq \inf_{\pi^0 \in \Pi^0} \mathbb{E}^{y_m} \left[ g(B^{\pi^0,0,0}(T), S^{\pi^0,0,0}(T)) \right] + \frac{1}{m}$$

$$\geq \mathbb{E}^{y_m}\Big[g(B^{\pi_m^0,0,0}(T), S^{\pi_m^0,0,0}(T))\Big]$$

for some $\pi_m^0 \in \Pi^0$. Also it easily see that $(B_{y_m}^{\pi_m^0,0,0}(T), S_{y_m}^{\pi_m^0,0,0}(T)) \to z$ a.s. as $m \to \infty$ after possibly passing to a subsequence. Hence Fatou's lemma gives $U_*(T,z) \geq g_*(z)$.
**4.** Thanks to the same argument as in Proposition 4 of Soner & Touzi[24] we obtain $U_*(T,\cdot) \geq \widehat{(g_*)}_0$ on $(0,\infty)^{n+1}$. □

**Lemma 5.6** *Let $\beta_0$ be a positive constant. Assume* (3.10). *Then, for any smooth test function $\varphi$ and local maximizer $s_0 \in \mathbf{R}_+^n$ of $(U^*(T,\beta_0,\cdot) - \varphi)$ on $\mathbf{R}_+^n$, we have*

$$\min\Big\{U^*(T,\beta_0,s_0) - h(\beta_0,s_0), \mathcal{H}_0(D\varphi(s_0))\Big\} \leq 0, \tag{5.5}$$

*where $h = g$ if $g$ is continuous and $h = \widehat{g}_0$ otherwise.*

Proof **1.** Let $z_0 \in (0,\infty) \times \mathbf{R}_+^n$ and $\psi$ be an $\mathbf{R}$-valued smooth test function on $\mathbf{R}_+^{n+1}$, and suppose that

$$0 = (U^*(T,\cdot) - \psi)(z_0) = \max_{\mathbf{R}_+^{n+1}}(U^*(T,\cdot) - \psi).$$

We first show

$$\min\Big\{U^*(T,z_0) - h(z_0), \mathcal{H}_0(D\psi(z_0))\Big\} \leq 0.$$

For this purpose, we assume $U^*(T,z_0) > h(z_0)$ and seek to show $\mathcal{H}_0(D\psi(z_0)) \leq 0$.

Since $h(z_0) - \psi(z_0) < U^*(T,z_0) - \psi(z_0) = 0$, we have

$$h(z) - \psi(z) \leq 2^{-1}(h(z_0) - \psi(z_0)) < 0, \qquad z \in A_{2\rho},$$

for sufficiently small $\rho > 0$, where $A_\rho$ is as in Lemma 5.3.

For any $m > T^{-1/2}$, we define $J_m := (T - m^{-2}, T)$ and

$$\psi_m(t,z) := \xi(z) + m(T-t) = \psi(z) + |z - z_0|^2 + m(T-t), \qquad (t,z) \in \overline{J}_m \times A_{2\rho}.$$

Let $\{(t_m,z_m)\}_m$ be a maximizing sequence of $(U^* - \psi_m)$ on $\overline{J}_m \times \partial_0 A_{2\rho}$, where $\partial_0 A := cl(\partial A \cap (0,\infty)^{n+1})$. Then, after passing to a subsequence, $z_m \to z_*$ as $m \to \infty$ for some $z_* \in \partial_0 A_{2\rho}$, and thus

$$\begin{aligned}\limsup_{m\to\infty} \sup_{\overline{J}_m \times \partial_0 A_{2\rho}} (U^* - \psi_m) &\leq \limsup_{m\to\infty}(U^*(t_m,z_m) - \psi(z_m)) - 4\rho^2 \\ &\leq U^*(T,z_*) - \psi(z_*) - 4\rho^2 \qquad \leq -4\rho^2.\end{aligned}$$

Since $U(T,z) \leq h(z)$, we have

$$\sup_{\partial_p(J_m \times A_{2\rho})}(U - \psi_m) \leq \max\Big\{\sup_{A_{2\rho}}(U(T,\cdot) - \psi), \sup_{\overline{J}_m \times \partial_0 A_{2\rho}}(U^* - \psi_m)\Big\}$$

$$\begin{aligned}
&\leq 2^{-1}\max\{h(z_0)-\psi(z_0),-\rho^2\} \\
&< 0 = U^*(T,z_0)-\psi(z_0) \\
&\leq \sup_{J_m^0\times A_\rho}(U^*-\psi_m)
\end{aligned}$$

for sufficiently large $m$. Hence if follows form Lemma 5.3 that

$$(\overline{J}_m\times A_{2\rho})\cap\mathcal{M}_0(\psi_m)^c\neq\emptyset \qquad\qquad \text{for sufficiently large } m.$$

Let $\{(t_m^*,z_m^*)\}_m\subset(\overline{J}_m\times A_{2\rho})\cap\mathcal{M}_0(\psi_m)^c$ and $z^*:=\lim_{m\to\infty}z_m^*\in A_{2\rho}$. Since

$$-\sup_{\pi^1\in K_1}\mathcal{G}^{D\psi_m(t_m^*,z_m^*),\pi^1}\psi_m(t_m^*,z_m^*)=m-\sup_{\pi^1\in K_1}\mathcal{G}^{D\xi(z_m^*),\pi^1}\xi(t_m^*,z_m^*)\to\infty \quad \text{as } m\to\infty,$$

then, we obtain

$$\begin{aligned}
&0\geq\lim_{m\to\infty}\mathcal{H}_0(D\psi_m(t_m^*,z_m^*))=\lim_{m\to\infty}\mathcal{H}_0(D\xi(z_m^*))=\mathcal{H}_0(D\psi(z^*)+2|z^*-z_0|) \\
&\to\mathcal{H}_0(D\psi(z_0)) \qquad \text{as } \rho\downarrow 0.
\end{aligned}$$

**2.** The similar arguments to Step 1 of Lemma 5.5 yields (5.5). □

**Lemma 5.7** *Assume that* $\gamma_0$ *in* (3.9) *is an element of* $int(K_0\cap\mathbf{R}_+^n)$ *and the conditions* (3.10)-(3.11) *hold. Then* $U^*(T,\cdot)\leq\widehat{g}_0$ *on* $(0,\infty)\times\mathbf{R}_+^n$.

*Proof* Fix arbitrary $\beta>0$. Let $h$ be as in Lemma 5.6. Define

$$u(s):=e^{U^*(T,\beta,s)},\qquad v(s):=e^{\widehat{g}_0(\beta,s)},\qquad \widetilde{h}(s):=e^{h(\beta,s)},\qquad s\in\mathbf{R}_+^n.$$

To prove $u\leq v$, we assume to the contrary that $2\zeta:=u(s_0)-v(s_0)>0$ for some $s_0\in\mathbf{R}_+^n$, and let us work towards a contradiction.

Let us introduce $\gamma:=\gamma_0+\rho\mathbf{1}_n\in int(K_0\cap\mathbf{R}_+^n)$ for sufficiently small $\rho>0$, and $\eta(s):=\exp(\gamma^\top s)$. For any $m>0$ we define

$$\varphi_m(s,s'):=u(s)-v(s')-\frac{m}{2}|s-s'|^2-\varepsilon\eta(s),\qquad M_m:=\sup_{\mathbf{R}_+^n\times\mathbf{R}_+^n}\varphi_m(s,s'),$$

where $\varepsilon>0$ is a small constant satisfying

$$M_m\geq u(s_0)-v(s_0)-\varepsilon\eta(s_0)=\zeta>0.$$

From (3.12), we have $u(s)\leq C'\eta(s)\exp(-\rho s^\top\mathbf{1}_n)$ for some constant $C'>0$. Hence $M_m=\varphi_m(s_m,s_m')$ for certain $s_m,s_m'\in\mathbf{R}_+^n$, and

$$\zeta+\frac{m}{2}|s_m-s_m'|^2+\varepsilon\eta(s_m)\leq u(s_m)-v(s_m'), \tag{5.6}$$

which provides that $\{s_m\}_m$ and $\{s_m'\}_m$ are located in a compact subset of $\mathbf{R}_+^n$. Therefore, after passing to a subsequence, $s_m\to s^*\in\mathbf{R}_+^n$. Further Lemma 3.1 in Crandall *et al.*[5]

gives

$$m|s_m - s'_m|^2 \to 0 \qquad \text{and} \qquad M_m \to u(s^*) - v(s^*) - \varepsilon\eta(s^*) \qquad \text{as } m \to \infty.$$

Let $p'_m := m(s_m - s'_m)$ and $p_m := p'_m + \varepsilon\gamma\eta(s_m)$. Then it follows from (5.5) that

$$\min\left\{u(s_m) - \widetilde{h}(s_m), u(s_m)\mathcal{H}_0\left(\frac{p_m}{u(s_m)}\right)\right\} \leq 0. \tag{5.7}$$

Since $\mathcal{H}_0(D\widehat{g}_0(\beta, s)) \geq 0,\ \ s \in (0,\infty)^n$, in the viscosity sense, we also have

$$\widehat{\mathcal{H}}_0\left(\frac{p'_m}{v(s'_m)} : s'_m\right) \geq 0,$$

where

$$\widehat{\mathcal{H}}_0(p:s) := \inf\Big\{\delta_0(q) - q^\top p \ :\ |q| = 1 \text{ and } \min_{1\leq j\leq n} q_j \mathbb{1}_{\{s_j=0\}} \geq 0\Big\}.$$

With the compactness of $\partial \mathrm{B}_1(0)$, we note

$$u(s_m)\mathcal{H}_0\left(\frac{p_m}{u(s_m)}\right) = u(s_m)\delta_0(q_m) - q_m^\top p_m$$

for some $q_m \in \partial\mathrm{B}_1(0)$. Define $q'_m := (q'_{m1}, \ldots, q'_{mn}) \in \partial\mathrm{B}_1(0)$ with $q'_{mj} := q_{mj}\mathbb{1}_{\{s'_{mj}>0\}} + |q_{mj}|\mathbb{1}_{\{s'_{mj}=0\}}$. Taking account of (3.1), (3.11) and (5.6), we obtain

$$\begin{aligned}
&u(s_m)\mathcal{H}_0\left(\frac{p_m}{u(s_m)}\right) - v(s'_m)\widehat{\mathcal{H}}_0\left(\frac{p'_m}{v(s'_m)} : s'_m\right)\\
&\geq \{u(s_m) - v(s'_m)\}\delta_0(q_m) - q_m^\top(p_m - p'_m) + \{\delta_0(q_m) - \delta_0(q'_m)\}v(s'_m) + (q'_m - q_m)^\top p'_m\\
&\geq \varepsilon\eta(s_m)\{\delta_0(q_m) - q_m^\top\gamma\} + \{\delta_0(q_m) - \delta_0(q'_m)\}v(s'_m) + 2m\sum_{j=1}^n q_{mj}^- s_{mj}\mathbb{1}_{\{s'_{mj}=0\}}\\
&\geq \varepsilon\eta(s_m)\mathcal{H}_0(\gamma)\\
&> 0.
\end{aligned}$$

Since $v \geq \widetilde{h}$, we derive the contradiction from (5.6)-(5.7):

$$\begin{aligned}
0 \geq \limsup_{m\to\infty}\big\{u(s_m) - \widetilde{h}(s_m)\big\} &\geq \limsup_{m\to\infty}\big\{u(s_m) - v(s'_m) + \widetilde{h}(s'_m) - \widetilde{h}(s_m)\big\}\\
&\geq \zeta + \limsup_{m\to\infty}\big\{\widetilde{h}(s'_m) - \widetilde{h}(s_m)\big\} \qquad = \zeta.
\end{aligned}$$

□

By Lemma 5.5&5.7, we have $\widehat{g}_0 = \widehat{(g_*)_0} \leq U_*(T,\cdot) \leq U^*(T,\cdot) \leq \widehat{g}_0$ under the conditions (3.2), (3.9)-(3.11), and $\gamma_0 \in int(K_0 \cap \mathbf{R}^n_+)$, as asserted. □

# Appendix

## A Arbitrage opportunities

Here we consider an absence of arbitrage opportunity. In this appendix, we shall say:

(i) For each $k \in \{0,1\}$, $\pi^k \in \Pi^k$ is an *arbitrage opportunity for the large investor* $\mathbb{I}^k$

$$\text{if} \qquad \mathbb{P}\{M^{\pi^k,\pi}(T) \geq 0\} = 1, \qquad \mathbb{P}\{M^{\pi^k,\pi}(T) > 0\} > 0 \qquad \text{for all } \pi^{1-k} \in \Pi^{1-k}.$$

(ii) $\{p^\pi,\ \pi \in \Pi\} \subset \mathcal{P}$ is an *arbitrage opportunity for a small investor*

$$\text{if} \qquad \mathbb{P}\{M^{p^\pi,\pi}(T) \geq 0\} = 1, \qquad \mathbb{P}\{M^{p^\pi,\pi}(T) > 0\} > 0 \qquad \text{for all } \pi \in \Pi.$$

$$\text{Here} \qquad M^{p,\pi}(t) := \int_0^t p(u)^\top dS^\pi(u), \qquad 0 \leq t \leq T, \quad p \in \mathcal{P}, \quad \pi = (\pi^0, \pi^1) \in \Pi.$$

For $\pi = (\pi^0, \pi^1) \in \Pi$, we shall denote by $\mathcal{A}_0(\pi)$ (resp. $\mathcal{A}_1(\pi)$) the set of all processes $p \in \mathcal{P}$ such that

$$\mathbb{P}\Big\{M^{p,\pi}(t) \geq -a^{p,\pi}, \quad \forall t \in [0,T]\Big\} = 1 \qquad \text{for some constant } a^{p,\pi} > 0$$
$$\left(\text{resp.} \qquad \mathbb{E}\left[\int_0^T |p(t)|^{2q} dt\right] < \infty \qquad \text{for some constant } q > 1\right).$$

We also define the sets $F^\pi(p), G^\pi \in \mathcal{B}[0,T] \otimes \mathcal{F}_T$ as

$$F^\pi(p) := \left\{(t,\omega) \in [0,T] \times \Omega \;\middle|\; \begin{array}{l} \bullet\ \sigma^\pi(t,\omega) diag[S^\pi(t,\omega)] p(t,\omega) = 0. \\ \bullet\ p(t,\omega)^\top diag[S^\pi(t,\omega)](b^\pi(t,\omega) - r^\pi(t,\omega)\mathbf{1}_n) > 0. \end{array}\right\},$$
$$G^\pi := \left\{(t,\omega) \in [0,T] \times \Omega \;\middle|\; b^\pi(t,\omega) - r^\pi(t,\omega)\mathbf{1}_n \notin Range[\sigma^\pi(t,\omega)^\top]\right\},$$

for $p \in \mathcal{P}$ and $\pi \in \Pi$. Then we have

**Proposition A.1** (i) *Suppose that* (2.5) *holds. Let $k \in \{0,1\}$ and $\pi^k \in \Pi^k$. If $\pi^k \in \mathcal{A}_0(\pi) \cup \mathcal{A}_1(\pi)$ for some $\pi^{1-k} \in \Pi^{1-k}$, then $\pi^k$ is not an arbitrage opportunity for the large investor $\mathbb{I}^k$.*

(ii) *Suppose that* (2.5) *holds. Let $\{p^\pi,\ \pi \in \Pi\} \subset \mathcal{P}$. If $p^\pi \in \mathcal{A}_0(\pi) \cup \mathcal{A}_1(\pi)$ for some $\pi \in \Pi$, then $\{p^\pi,\ \pi \in \Pi\}$ is not an arbitrage opportunity for a small investor.*

(iii) *Fix $k \in \{0,1\}$. Assume that for all $\pi^{1-k} \in \Pi^{1-k}$ there exists $\pi^k \in \Pi^k$ such that* (Leb.$\otimes$ $\mathbb{P}$)$\{F^\pi(\pi^k)\} > 0$ *and $\pi^k_* := \pi^k \mathbb{1}_{F^\pi(\pi^k)} \in \Pi^k$. Then $\pi^k_*$ is an arbitrage opportunity for the large investor $\mathbb{I}^k$.*

(iv) *If* (Leb.$\otimes\mathbb{P}$)$\{G^\pi\} > 0$ *for all $\pi \in \Pi$, then there is an arbitrage opportunity for a small investor in market.*

*Proof* We adapt the arguments in Karatzas [15, §0.2] to our market model.
**1.** Let $\pi \in \Pi$ and $p \in \mathcal{A}_0(\pi) \cup \mathcal{A}_1(\pi)$. In case $p \in \mathcal{A}_1(\pi)$, we know from (2.8) that $\{Z_{\theta^\pi}(t)M^{p,\pi}(t),\ 0 \le t \le T\}$ is a $\mathbb{P}$-martingale. In case $p \in \mathcal{A}_0(\pi)$, we observe

$$\begin{aligned}
0 &\le Z_{\theta^\pi}(t)\{M^{p,\pi}(t) + a^{p,\pi}\} \\
&= \int_0^t Z_{\theta^\pi}(u)\left[\sigma^\pi(u) diag[S^\pi(u)]p(u) + \{M^{p,\pi}(u) + a^{p,\pi}\}\theta^\pi(u)\right]^\top dW(u),
\end{aligned}$$

which means that the process $\{Z_{\theta^\pi}(t)M^{p,\pi}(t),\ 0 \le t \le T\}$ is a $\mathbb{P}$-supermartingale. Hence we obtain $\mathbb{E}[Z_{\theta^\pi}(T)M^{p,\pi}(T)] \le 0$, and thus the condition $\mathbb{P}\{M^{p,\pi}(T) \ge 0\} = 1$ implies $M^{p,\pi}(T) = 0$ a.s. Therefore we have the assertions (i) and (ii).
**2.** Suppose that the assumption in (iii) is satisfied. Then we know that $\pi_*^k$ is an arbitrage opportunity for the large investor $\mathbb{I}^k$ by virtue of

$$M^{\pi_*^k,\pi}(T,\omega) = \int_0^T |\pi^k(t,\omega)^\top diag[S^\pi(t,\omega)](b^\pi(t,\omega) - r^\pi(t,\omega)\mathbf{1}_n)| \mathbb{1}_{F^\pi(\pi^k)}(t,\omega)dt.$$

**3.** In case (Leb.$\otimes\mathbb{P}$)$\{G^\pi\} > 0$ for all $\pi \in \Pi$, we easily see that for each $\pi \in \Pi$ there exists a $p^\pi \in \mathcal{P}$ such that (Leb.$\otimes\mathbb{P}$)$\{F^\pi(p^\pi) \cup F^\pi(-p^\pi)\} > 0$. For each $\pi \in \Pi$, let us define

$$\widehat{p}^{\,\pi} := \mathrm{sgn}\{(p^\pi)^\top diag[S^\pi](b^\pi - r^\pi\mathbf{1}_n)\} p^\pi \mathbb{1}_{F^\pi(p^\pi)\cup F^\pi(-p^\pi)}\ .$$

Clearly, $\{\widehat{p}^{\,\pi},\ \pi \in \Pi\}$ is then an arbitrage opportunity for a small investor. □

## B Case of depending on proportions of wealth

Let $\mathcal{Q}^k$ $(k = 0, 1)$ be subsets of $\mathcal{P}$ such that $0 \in \mathcal{Q}^k$, and we denote $q = (q^0, q^1) \in \mathcal{Q} := \mathcal{Q}^0 \times \mathcal{Q}^1$. Here $q^k(t) = (q_1^k(t), \ldots, q_n^k(t))$ represents the proportions of $\mathbb{I}^k$'s wealth invested in the corresponding stocks at time $t \in [0, T]$, for each $k \in \{0, 1\}$.

In this subsection we assume that the coefficients $r^\pi, b^\pi$ and $\sigma^\pi$ of market depend on merely the pair $q = (q^0, q^1)$ of the portfolio-proportion processes. Then the discounted wealth process $X_k^{x_k,q}(\cdot)$ of the investor $\mathbb{I}^k$ evolves according to the equation

$$\frac{dX_k(t)}{X_k(t)} = q^k(t)^\top \Big\{(b^q(t) - r^q(t)\mathbf{1}_n)dt + \sigma^q(t)^\top dW(t)\Big\}, \qquad 0 \le t \le T$$

with an initial capital $X_k(0) = x_k \in \mathbf{R}$, $k = 0, 1$.

We also assume that the contingent claim $\{BC, \mathcal{T}\} = \{B^qC^q, \mathcal{T}^q\}$ only depends on the pair $q = (q^0, q^1)$. Then, by analogy with Theorem 2.1, we have

**Theorem B.1** *The minimal hedging costs are expressed as*

$$\begin{aligned}
h_{up} &= \sup_{q^1\in\mathcal{Q}^1} \inf_{q^0\in\mathcal{Q}^0} \sup_{\tau\in\mathcal{T}^q} \sup_{\nu\in\mathcal{D}} \mathbb{E}_\nu\left[\frac{C^q(\tau)}{X_0^{1,q}(\tau)}\right], \\
h_{low} &= \lim_{m\to\infty} \inf_{q^0\in\mathcal{Q}^0} \sup_{q^1\in\mathcal{Q}^1} \sup_{\tau\in\mathcal{T}^q} \inf_{\nu\in\mathcal{D}} \mathbb{E}_\nu\left[\frac{C^q(\tau)}{X_1^{1,q}(\tau)} \wedge m\right],
\end{aligned}$$

*respectively. Moreover, if* $\mathbb{E}[(C^q(\tau)/X_1^{1,q}(\tau))^p] < \infty$ *for any* $q \in \mathcal{Q}$, $\tau \in \mathcal{T}^q$ *and some constant* $p = p(q,\tau) > 1$, *then*

$$h_{low} = \inf_{q^0 \in \mathcal{Q}^0} \sup_{q^1 \in \mathcal{Q}^1} \sup_{\tau \in \mathcal{T}^q} \inf_{\nu \in \mathcal{D}} \mathbb{E}_\nu \left[ \frac{C^q(\tau)}{X_1^{1,q}(\tau)} \right]$$

*Proof* The proof is similar to that of Theorem 2.1. Thus we omit it here. □

**Remark B.2** Let us consider the model:

- $b^q$ and $\sigma^q$ are independent of the pair $q = (q^0, q^1)$ of portfolio-proportion processes,
- $r^q(t) = r(t) + \rho(t)\mathbb{1}_{\{q^0(t)^\top \mathbf{1}_n > 1\}}$, $0 \le t \le T$, for certain bounded, positive, $\mathbb{F}$-progressively measurable processes $r, \rho$ which are independent of $q$.

Since $X_0^{x_0,q}(\cdot) \ge 0$ a.e. for $q \in \mathcal{Q}$ and $x_0 \ge 0$, this model describes the market where the interest rate for borrowing is higher than the interest rate for investing. This model was studied by Cvitanić & Karatzas[7] and Cvitanić & Ma[9].

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